Study of Time-Dependent Queuing Models of the National Airspace System

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Abstract

Queuing models provide an attractive and highly-efficient alternative to simulation for quantifying traffic flow efficiency. Stationary Markovian queuing models in which both inter-arrival times and service times are exponentially distributed have been studied by the National Airspace System (NAS). However, stationary queues cannot account for peaks and valleys in demand that are commonly observed in the NAS. Thus time-dependent Markovian queuing models, which aim to capture the variation in demand during a day, have been studied. Furthermore, statistical analysis of real traffic data reveals that inter-arrival times and service times do not follow exponential distributions. As a subclass of phase-type distributions, Coxian distributions with the advantage of closely approximating any distribution without violating the Markov property, have gained special importance on research in queuing systems. In this research, time-dependent Coxian queuing models $C_{m(t)}(t)/C_k/s/s$ for modeling the en route phases of flight are developed as well, which are approximated by a piecewise constant Coxian inter-arrival time distribution and a time-invariant Coxian service time distribution. Both arrival rates and service rates are calibrated from data extracted from high-fidelity simulation runs driven by actual flying data. The number of aircraft in the system is regarded as a measure of the accuracy of queuing performance. Comparison results between time-dependent Markovian and Coxian queuing models are given in this paper. This study shows that time-dependent Markovian queues could capture the variation in demand as well as Coxian queues, with the advantage of mathematical and computational tractability.

Keyword: *National Airspace* *Time-dependent queuing models* *Coxian distribution* *Markovian queue* *Queuing performance*

1 Introduction

Air transportation in the US system has dramatically changed in the past few decades. The National Airspace System (NAS) has increasingly become congested. Air traffic delay problems have resulted in huge economic loss to both passengers and industries. According to the (NASA, 2008a), in 1998, airline delays in the U.S. cost industries and passengers $4.5 billion. Kim and Hansen (2013) and Santos and Robin (2010) has identified significant factors that could account for delays and analyzed resulting flight delays at an airport caused by demand (capacity) changes with constant capacity (demand). Air traffic delay problems can be improved by two approaches (Terrab and Odoni, 1993): (1) expanding system capacity in the form of either building new infrastructure, such as runways and airports, or new capacity enhancing technology; (2) traffic flow management, traffic flow management studies include slot control (Swaroop et al., 2012), congestion pricing (Daniel, 1995), and air traffic delay studies (Koopman, 1972; Pyrgiotis et al., 2013; Terrab and Odoni, 1993). The aforementioned air traffic delay studies, such as air traffic delay modeling in the air terminal (Koopman, 1972) or delay propagation modeling within an airport network (Pyrgiotis et al., 2013), mainly focus on using queuing modeling methods to predict aircraft delay given inter-arrival time, service time, and capacity information. Usually, capacity is assumed to be reduced, which causes the predicted delays.

Different from delay prediction queuing modeling (Koopman, 1972; Pyrgiotis et al., 2013), this research focuses on using queuing modeling to quantify the relationship between the number of aircraft in the NAS and time-dependent inter-arrival time and service time. As precision of navigational instruments improve, aircraft are allowed to fly faster and closer to each other. As a result, capacity in the NAS is expected to increase. Consequently, models that can accurately and quickly provide measures of future air traffic scenarios in which more aircraft are flying in the airspace are vital to an efficient and effective air traffic management system. NAS has been divided into four spatial resolution, namely airspace, center, sector, and cell level from coarse to fine resolution (Menon et al., 2008). In this research, we model flights in the en route phase in cells defined by 1.5-degree-by-1.5-degree (1.5x1.5) airspace grid above one of five major airports, ATL, DFW, JFK, LAX and ORD, which account for a large percentage of air traffic in the NAS. Queuing performance is analyzed for these five cells with data from June 1 to 7, 2007. The methodology for cell-level spatial resolution can be used for other resolutions as well.

Air-traffic flow simulation software such as the Future ATM Concepts Evaluation Tool (FACET) can be used to quantify traffic flow efficiency. By using actual air traffic data from the Federal Aviation Administration (FAA), FACET can be used to analyze congestion patterns of different cells of the airspace by propagating the trajectories of proposed flights forward in time (FAA, 2008). However, the time consuming characteristic of FACET is not amenable for conducting rapid trade studies needed to evaluate future NAS scenarios. By contrast, modeling the air traffic flow by queuing models could provide quantitative information about the effects of better precision and navigation tools and corresponding increases in capacity on operations of the NAS in an effective and time-efficient manner (Long et al., 1999).

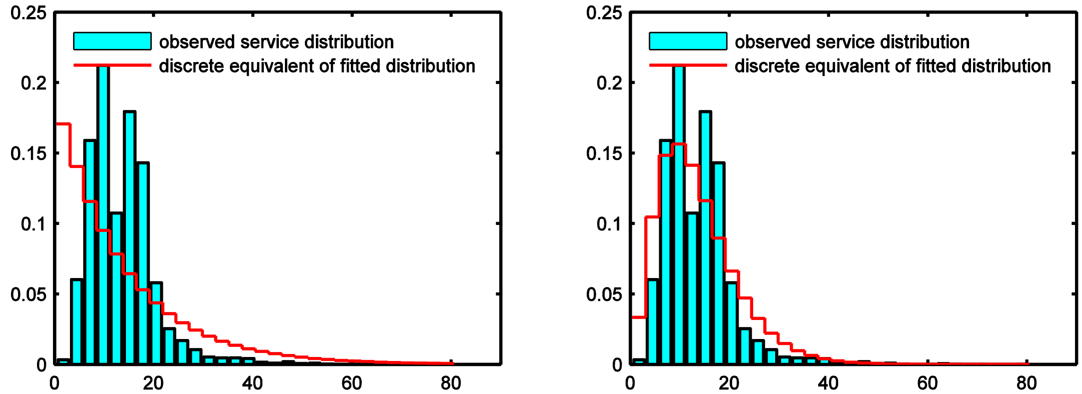
Queuing theory is first known from the work of A. K. Erlang of the Copenhagen Telephone Company in 1900s. Nowadays, it is used widely to analyze computer systems (Kobayashi and Konheim, 1977; Phillips and Kokotovic, 1981; Sauer, 1981), communication systems (Bhulai and Koole, 2003; Koole and Mandelbaum, 2002; Krieger et al., 1990) and transportation systems (Heidemann, 1991, 1994; Jain and Smith, 1997; Pearce, 1967; Vandaele et al., 2000). Yet, most research about queuing models focuses on stationary queuing models. However, in most real applications, the arrival rate is time-dependent. For time-dependent, even if for moderately time dependent Markovian queuing systems, research has shown that results of stationary models are quite inaccurate (Green et al., 1991).

The probability distribution of inter-arrival times and service times could almost take any form in empirical data. However, the assumed distribution should be realistic, such as frequently used exponential distribution, so that the queuing model could provide reasonable predictions while at the same time being tractable (Tandale et al., 2008). Since Coxian distributions have the advantage of approximating any arbitrary distribution without violating the Markov property, it can fit both inter-arrival and service times in the queuing modeling. In this study, the *arrival rate* is the number of aircraft entering a cell per unit time, which fluctuates widely during a day. In order to capture the variation in the arrival process, both time-dependent Markovian queuing models $M(t)/M/s/s$ and time-dependent Coxian queuing models $C_{m(t)}(t)/C_k/s/s$ are developed. Goodness of fit chi-squared tests for both exponential distributions and 3-phase Coxian distributions of service times at the aforementioned five cells are in Table **1**, which show that Coxian distribution chi-squared values are smaller than those of exponential distributions, though both types of distributions are rejected given a significance level $\alpha = 0.05$. Hence, Coxian distributions fit the data better. Results of fitting both the exponential distribution and a 3-phase Coxian distribution to data of service times from June 1 to 7, 2007, at LAX cell-level space are shown in Figure 1 separately. The figure consistently shows the Coxian distribution fits the data better. Thus, for exploring more accurate queuing performance measures, time-dependent Coxian queuing models $C_{m(t)}(t)/C_k/s/s$ are developed.

chi-squared value	ATL	DFW	JFK	LAX	ORD
exponential	6368	13788	10414	8728	12972
Coxian	4044	6813	6266	1959	6523

Table 1. Chi-squared value for service time distribution at five cells

In the remainder of Section 1, an overview of flight profiles and time-dependent queuing models is introduced. Section 2 describes the models used in this research, overviews the Coxian distribution and solutions for $C_m/C_k/s/s$ models, and presents the methodology of solving $C_{m(t)}(t)/C_k/s/s$ queues. In Section 3, both $M(t)/M/s/s$ and $C_{m(t)}(t)/C_k/s/s$ queuing models are validated by data extracted from FACET. In addition, comparison between steady state approximation and expected transient state approximation results are shown. Conclusions and discussions are given in Section 4.



Exponential distribution　　3-phase Coxian distribution

Figure 1. Service time distributions

1.1 Research Scope

An aircraft experiences several different phases during each flight, as shown in Figure 2. Different phase of flight can be modeled by different queuing systems separately (Long et al., 1999; Menon et al., 2008). This research only focuses on modeling the en route phase of flight. Menon et al. (2008) uses $M/M/\infty$ queues to model the en route phase of flight since typically the available en route capacity is much larger than the demand under normal conditions. Long et al. (1999) uses $M/E_k/N/N+q$ to model en route phases of flight and allow up to q aircraft to be delayed (forced to wait) due to speed changes or vectoring. However, in this research, we assume that speed adjusting or vectoring aircraft are modeled in service. Here, a *server* is defined as the airspace an individual aircraft occupies in a cell. Consequently, we use $M(t)/M/s/s$ or $C_{m(t)}(t)/C_k/s/\mathrm{s}$, which constrain NAS capacity and assume all flying aircraft are being served simultaneously. These models, which are referred to as *pure loss models*, are similar to those in roadway segments modeling (Jain and Smith, 1997). Although aircraft in pure loss models can theoretically be blocked when traffic is at capacity, blocking is atypical under normal conditions (Menon et al., 2008). The four factors of the developed queuing models are: (1) *inter-arrival time distribution*, which is defined as the time interval between two successive aircraft entering into a cell; (2) *service time distribution*, which is defined as the duration of time from when an aircraft enters a cell until it leaves the cell; (3) *number of servers in the system*, which is the number spaces individual aircraft occupy in a cell; (4) *system capacity*, which is the maximum number of aircraft allowed to fly in a cell. In this research, the inter-arrival time distribution is time-dependent, the service time distribution is time invariant, and the number of servers and the system capacity are equal as in a pure loss model.

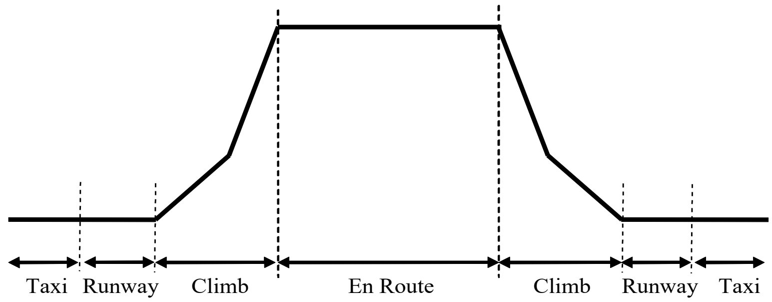


Figure 2. Flight profiles (Menon et al., 2008; NASA)

1.2 Overview of Time-Dependent Queuing Models

For time-dependent queuing models, solving exact numerical solutions is computationally cumbersome (Clark, 1981; Koopman, 1972; Rider, 1976; Rothkopf and Oren, 1979). Several approximation approaches, such as Pointwise Stationary Approximation (PSA is obtained by taking the expectation of the formula for stationary performance measure with an instant arrival rate at each time point as an input parameter) (Grassmann, 1983; Green et al., 1991; Rolski, 1986; Whitt, 1991), the Stationary Independent Period by Period (SIPP) approach (Green et al., 2001) and the Stationary Backlog Carryover (SBC) approach (Stolletz, 2008), Surrogate Distribution Approximation (SDA) (Gelenbe and Rosenberg, 1990; Rolski, 1987), Diffusion Approximation (Duda, 1986; Newell, 1968a, b, c) for computing solutions of time-dependent queuing models, are prevalent.

For strongly a time-dependent arrival process as in this research, segmenting the entire time period into a series of individual segment, like SIPP and SBC approaches, is a practical analytical approach for the current research situation. In each segment, the arrival rate is approximated by the average arrival rate during that segment, which is used as input to a stationary queuing model. The approximation approach used in this research is also called *Piecewise Constant Coxian (Markovian) Queues*. As in the SBC approach, this research also assumes that different time periods are not independent from each other. In two successive time periods, the same number of aircraft may be represented by different state space probability vectors. Consequently, a projection algorithm is developed to shift the state probability vector from one period to the next period. In contrast to the SBC and SIPP approaches, this research assumes that in each time period, steady state might not be achieved. Thus, the number of aircraft in the system (a cell) as a performance measure for both $M(t)/M/s/s$ and $C_{m(t)}(t)/C_k/s/s$ models, is calculated by averaging transient solutions over each time period. The number of aircraft in the system can be 0, …, s, and as a performance measure, is compared between these two types of queuing models. Both queuing models are validated by data extracted from FACET, which is driven by empirical flying data. Results show that the Markovian queue performs as well as the Coxian queue with the advantage of mathematical and computational tractability. Moreover, expected transient solutions are more accurate than steady state solutions for both types of queuing models.

1.3 Contribution

This paper makes the following contributions: (1) this research develops time-dependent queuing models that can provide measures of future NAS situations in which capacity is increased due to improvements in precision and navigation equipment. The methodology of this research can be used for other time-dependent queuing modeling situations as well; (2) a practical approach of using piecewise constant $C_m/C_k/s/s$ to approximate time-dependent queuing models $C_{m(t)}(t)/C_k/s/s$ is developed. Because the arrival process changes in each time period of the $C_{m(t)}(t)/C_k/s/s$ queue, the state spaces are different in different time periods. Consequently, this research develops a practical linear programming projection algorithm that shifts a probability state vector from a state space in one time period to that of the next period; (3) our results show that $M(t)/M/s/s$ queuing models perform as well as $C_{m(t)}(t)/C_k/s/s$ and $M(t)/M(t)/s/s$ models with the advantage of mathematical and computational tractability over the former set of models.

2 Methodology

2.1 Parameters of Pure Loss Model for NAS

As mentioned in Section 1.1, pure loss models are appropriate for modeling the NAS. A $C_{m(t)}(t)/C_k/s/s$ model and its calibration is introduced in this section.

2.1.1 Data Source and Assumptions

FACET is a high-fidelity traffic flow simulation software package, which is driven by empirical flying data from the FAA (NASA, 2008b). FACET does not simulate ascent or descent below 10,000 feet. Thus a 1.5x1.5 cell data extracted from FACET does not include ascent or descent below 10,000 feet. Though a 1.5x1.5 cell can possibly include part of the trajectory in TRACON, we can regard a 1.5x1.5 cell data as mainly including the en route flight segment data. All data used in this research are extracted from FACET by taking a record for each flying aircraft. The record includes an aircraft's ID number, location (which cell they are currently in) and current time. Records are snapshots taken every 30 seconds to check whether a state changed (landing or transiting to other cell location) during the aircraft flying process. If there is a state change, the record will be accordingly updated. Thus, according to an aircraft's record, the service time (time used for an aircraft to fly across a cell) and inter-arrival time (time interval between two successive aircraft entering into a cell) and number of aircraft in a cell can be calculated based on the record easily. In this research, data are from five 1.5x1.5 cells located above the major airports ATL, DFW, JFK, LAX, and ORD from June 1 to 7, 2007. The sizes of the five cells, which can be approximated in nautical miles (Briney, 2014), are shown in **Table 2**, where w is the distance (nautical miles: nmi) between longitude and h is the distance between latitude.

distance	ATL	DFW	JFK	LAX	ORD
w (nmi)	75.04	76.32	67.88	75.04	67.88
h (nmi)	89.84	89.82	89.95	89.84	89.95

Table 2: the size of five 1.5x1.5 cells in nautical miles

Calibration of model parameters is based on following assumptions:

Assumption 1: It is possible to un-truncate the arrival data.

Theoretically, an arrival event can occur in 30 seconds. This research assumes arrival data is un-truncated.

Assumption 2: The set of times representing fundamental changes in the inter-arrival distribution is 24 one-hour time periods.

As mentioned in Section 1.2, the method of segmenting the entire time horizon into a series of individual segments is employed in this research. This research assumes that in every hour the arrival rate can be approximated by a time invariant random variable, which may be different from the next hour segment.

In addition to Assumptions 1 and 2, for much of this paper, we assume that the distribution of service times does not change and remains stable for the entire time horizon. Service time though can be affected by different aircraft type and by random events such as rerouting. However, as long as aircraft type, flight routes, and rerouting do not have strong time-dependent patterns, the assumption of time invariant service time distributions is reasonable. Nonetheless, we also investigate $M(t)/M(t)/s/s$ queuing models in which service time distributions may change every six hours in Section 3.3.

2.1.2 Capacity and Number of Service Channels

The *capacity* of a single airspace cell is defined by the maximum number of aircraft allowed to operate in a cell in a given time. However, calculating capacity is not well defined for a particular cell, and the queuing modeling approach of the NAS in this paper could be done with alternative capacity estimates. In this research, capacity is passed as a known parameter from upstream research, and the systematic procedure for calculating cell capacity is described in Menon et al. (2008). A summary of the calculation procedure is as follows: Let s be the capacity of the given cell, r be the maximum acceptable conflict rate of the cell, Q be the volume of the cell, D_h be the minimum horizontal separation distance, D_v be the minimum vertical separation distance, and $E[V_{rel}]$ be the mean relative speed. The capacity of the airspace is estimated by equation (1):

$$s = \sqrt{rQ/(2D_h D_v E[V_{rel}]) + 0.25} - 0.5 \tag{1}$$

Here, the parameters D_h, D_v, $E[V_{rel}]$ and Q can be computed directly from cell geometry and traffic parameters, and commonly accepted values are $D_h = 5$ nmi, $D_v = 0.165$ nmi, and $E[V_{rel}] = 483$ kt. Conflict rate r can be estimated based on the monitor alert parameter (MAP) for a sector in the NAS, typically $r = 0.56$ conflicts/hour.

In extreme weather, the capacity of some cells (and airports) in the NAS could drop significantly (Bertsimas and Patterson, 1998). However, improved precision of navigational equipment allows aircraft to fly closer together and increase capacity. The capacity in a single cell is denoted by s in the following sections of this paper. The space of individual aircraft occupied in the airspace represents one queuing "server".

2.1.3 Inter-Arrival Time and Arrival Rate

Inter-arrival time is defined as the time interval between two successive aircraft entering into an individual cell. If in 30 seconds there are n (n>1) aircraft entering into the cell, those aircraft are assumed to have arrived into the cell in equal time periods of length $30/n$. With inter-arrival time data, we fit Exponential distributions and Coxian distributions. Since arrival rate varies during a day, a time-dependent arrival rate is assumed.

Using seven days of FACET data, MATLAB code was developed to extract arrivals that include both external sources and aircraft arriving from other cells within the network directly from FACET. Let $X_1,\ldots,X_l$ be a set of independent Coxian random variables for the inter-arrival time at different periods throughout the day. In this study, we assume l represented a one-hour time period. Then, the time-dependent inter-arrival time is approximated by *a time dependent piecewise constant Coxian random variable* given by equation (2):

$$X(t) = \begin{cases} X_1 & t_0 \le t < t_1 \\ X_2 & t_1 \le t < t_2 \\ \vdots & \vdots \\ X_l & t_{l-1} \le t < t_l \end{cases} \tag{2}$$

For each time period, fitting an inter-arrival time to a Coxian distribution, the parameters of arrival rate μ_i in each phase and the continuation probabilities a_i is fitted by the Expected Maximum Likelihood Estimation (EM) algorithm (Asmussen et al., 1996). The service time Coxian parameters are also found by the EM algorithm. To keep the EM algorithm computationally tractable, both inter-arrival and service time distributions are limited to three phases.

2.1.4 Service Time and Service Rate

Service time is defined as the time an aircraft takes to cross a single cell. For example, FACET records an aircraft entering into a cell at time t_1 and leaving the cell at time t_2, so service time is calculated by $t_2 - t_1$. Although service times are generally assumed to be stationary, we investigate queuing models in which the service distribution is time dependent.

2.2 Background on Coxian Queues

2.2.1 Overview of Coxian Distribution

A *phase-type distribution* describes a random time taken for a continuous time Markov process to reach an absorbing state, where only one absorbing state exists and the stochastic process starts at a transient state. A phase-type distribution can be generalized to include many types of continuous distributions, such as Exponential distributions, Erlang distributions, Hypoexponential distributions, etc. With the advantage of approximating any non-negative continuous distribution, a phase-type distribution has the Markovian (memoryless) property. However, its generality can be problematic since it is overparameterized and parameter estimation is difficult. As a special case of a phase-type distribution, the Coxian distribution has the advantage of closely approximating any arbitrary nonnegative distribution and can overcome this problem to some extent. The Coxian distribution is a generalization of the Hypoexponential distribution in which a sequence of transient k-1 states can enter into the absorbing state. For a k-phase Coxian distribution, only $2k$-1 parameters need to be estimated (Cox, 1955; Perros, 1994), see Figure 3.

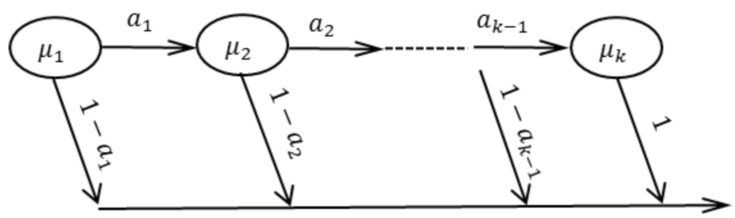


Figure 3. A k-phase Coxian distribution (Perros, 1994)

Therefore, the Coxian distribution is widely used in applications, such as in the health-care industry (Faddy, 1995; Faddy and McClean, 1999; Marshall and McClean, 2004). The method of fitting empirical data to a phase-type distribution has also been done by researchers. Popular methods are Maximum Likelihood Estimators (Bobbio and Cumani, 1992; Bobbio and Telek, 1994), the Expectation-Maximization algorithm (Asmussen et al., 1996; Dempster et al., 1977), moment matching (Johnson, 1993; Schmickler, 1992) and other methods (Esparza et al., 2010; Fackrell, 2009; Marshall and Zenga, 2012).

2.2.2 Coxian Queuing Model

As mentioned in Section 1.2, exact transient solutions of time-dependent queuing models are very cumbersome. For analyzing the transient behavior of a time-dependent $C_{m(t)}(t)/C_k/s/s$ queue, the transient performance analysis of a time invariant $C_m/C_k/s/s$ queue is required. An $M(t)/M/s/s$ queue uses similar but much simpler analysis than a $C_{m(t)}(t)/C_k/s/s$ queue, so we only present an analysis for $C_{m(t)}(t)/C_k/s/s$ queues. The Champman-Kolmogorov forward equation based state enumeration is used for the solution process if the inter-arrival and service time distributions are not exponential. The matrix notation for the Champman-Kolmogorov forward equation is as follows:

$$\dot{x}(t) = Qx(t) \tag{3}$$

where $x(t)$ is a probability vector in which the value $x_i(t)$ represents the probability of the system being in state i where i = 1, …, n. Q is an $n \times n$ matrix called the *infinitesimal generator matrix* or the *transition rate matrix* for a Markov process. The vector $\dot{x}(t)$ represents the derivative of $x(t)$ with respect to t.

By integrating $\dot{x}(t) = Qx(t)$ over each time period, with the normalizing condition equation $\sum_{i=1}^{n} x(i) = 1$, the transient solution to the Markov process can be solved given an initial state probability and rate matrix *Q*. Using the matrix exponential e^{Qt}, the time invariant system solution can be obtained by taking the limit as *t* approaches infinity.

The analysis of $C_m/C_k/s/s$ in each time period is nearly the same as the analysis of a queue system $C_m/C_k/s$ discussed in Senguputa et al. (2010). A $C_m/C_k/s/s$ queuing system can be presented as Figure 4, where $\mu_{G1}, \mu_{G2}, \ldots, \mu_{Gm}$ are the arrival parameters of an *m*-phase Coxian distribution, and $a_{G1}, a_{G2}, \ldots, a_{Gm}$ are the transition probabilities of the arrival process. We refer to this as a *generator*.

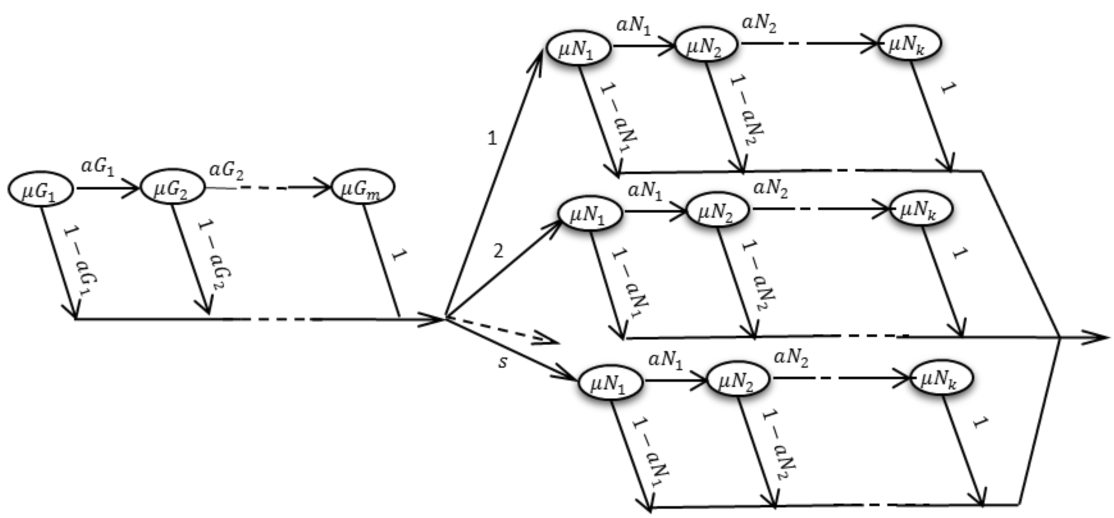


Figure 4. $C_m/C_k/s/s$ queuing system

Furthermore, service nodes of $C_m/C_k/s/s$ queue can be represented by a *k*-phase Coxian, with service parameters $\mu_{N1}, \mu_{N2}, \ldots, \mu_{Nk}$, and transition probabilities $a_{N1}, a_{N2}, \ldots, a_{Nk}$.

In this study, the $C_m/C_k/s/s$ queuing system state can be defined by: (1) the phase of the arrival process or generator, (2) the number of items in service, and (3) the number of servers in each phase. Let *a* be the phase of the generator, let *b* be the number of items in service, and let $c_1, c_2, \ldots, c_k$ be the number of servers in phases 1, 2,…, and *k*, respectively. The state of the system can be represented by the sequence $a{:}b(c_1, c_2, \ldots, c_k)$. The total number of servers in phase 1, 2, …, and *k* at a service node can be given as solutions to equation (4):

$$\sum_{i=1}^{k} c_i = b \qquad b = 0, 1, 2, \ldots, s \tag{4}$$

2.2.3 State Transitions

Let $N(s)$ be the total number of states for a $C_m/C_k/s/s$ queuing system. Then $N(n)$ is described in Senguputa et al. (2010):

$$N(s) = m\binom{s+k-1}{k} \tag{5}$$

The dimension of the transition rate matrix Q is $N(s) \times N(s)$ in which the $(i,j)^{\text{th}}$ element, q_{ij}, represents the rate of transition from the j^{th} state to i^{th} state. The transitions of a $C_m/C_k/s/s$ queue described as sequence $a{:}b(c_1, c_2, \ldots, c_k)$ above can be defined as the following events: 1. a phase change in the arrival node or generator; 2. an aircraft arrival into a service node (or departure from the generator); 3. an aircraft departure from the service node; 4. a phase change in the service node. For a $C_2/C_2/3/3$ queuing system, the dimension of transition rate matrix Q is 20×20, the rate of transition from state 1:1(1,0) to 2:1(1,0) is $\mu G_1 \times aG_1$; to 1:2(2,0) is $\mu G_1 \times (1 - aG_1)$; to 1:0(0,0) is $\mu N_1 \times (1 - aN_1)$; to *1:1(0,1)* is $\mu N_1 \times aN_1$ (notation is same as in Figure 4).

2.3 Time-Dependent Coxian Queues

In this section, the method of using a *Piecewise Constant Coxian Queue* to approximate a $C_{m(t)}(t)/C_k/s/s$ queue is described. The transient state probability vector was determined by integrating the Champman-Kolmogorov forward equation $\dot{x}(t) = Qx(t)$ over each the time period. The approach to determine average measures of the queues is discussed. Furthermore, a linear programming projection method was developed to shift the probability vector from one period to the next period. The number of aircraft as an average measure can be obtained for each time period, which is the metric to measure the queuing performance in this research.

2.3.1 Probability State Vector Determination

The transition matrix $Q(t)$ can be written as given in equation (6) due to the piecewise time-dependent inter-arrival time of a $C_{m(t)}(t)/C_k/s/s$ queuing model.

$$Q(t) = \begin{cases} Q_1 & t_0 \le t < t_1 \\ Q_2 & t_1 \le t < t_2 \\ \vdots & \vdots \\ Q_l & t_{l-1} \le t < t_l \end{cases} \tag{6}$$

This shows that the transition rate matrix varies, which can lead to the steady state probabilities not existing. However, we still would like to calculate certain average measures based upon the probabilities of each state over the time horizon using equation (7), where $M(x)$ is a measure based upon the state probability vector x, and $\overline{M_\iota}$ is the average of the measure over time periods $[t_{i-1}, t_i)$.

$$\overline{M_\iota} = \frac{\int_{t_{i-1}}^{i} M(x(t))dt}{t_i - t_{i-1}} \tag{7}$$

Equation (7) can be rewritten to determine the average state probability vector $\overline{x_\iota}$ for each time period $t_i, \forall i = 1, \dots, l$ and an average measure $\overline{M_\iota}$ for each time period using equation (8) due to the linearity of many standard queuing model measures with respect to the state probability vector.

$$\overline{M_\iota} = M(\overline{x_\iota}) \qquad (8)$$

Calculating $\bar{x}_i$ for each time period $t_i, \forall i = 1, \dots, l$ will be discussed. By integrating numerically using ODE45 in MATLAB, equation (9) shows how to calculate the state probability vector over time as follows:

$$x(t) = e^{Q_i(t - t_{i-1})} x(t_{i-1}), \qquad \forall t \in [t_{i-1}, t_i), \forall i = 1, \dots, l \qquad (9)$$

For a sufficiently small $\varepsilon > 0$, $x(t_i - \varepsilon)$ and $x(t_i)$ represent probability state vectors of two different state spaces at a given time t_i. Because this can lead to a significant calculation complication, $x(t_i)$, $\forall i = 1, \dots, l-1$ will be redefined by two different vectors. Let $x^+(t_i)$ and $x^-(t_i)$ be the state probability vectors at time t_i in state spaces associated with periods $[t_i, t_{i+1})$ and $[t_{i-1}, t_i)$, respectively for each time $t_i, \forall i = 1, \dots, l-1$. Variables $x(t_0)$ and $x(t_l)$ are redefined as $x^+(t_0)$ and $x^-(t_l)$, as shown in equation (10), to further specify notation.

$$x^-(t_i) = e^{Q_i(t_i - t_{i-1})} x^+(t_{i-1}), \quad \forall i = 1, \dots, l \qquad (10)$$

An algorithm to calculate the probability vector for each time period $\overline{x_\iota}, \forall i = 1, \dots, l$ is given by the following steps.

Step 1: Let $i = 1$ and $x^+(t_0)$ be given. At the very beginning point in time t_0, the system is not empty. In this research, $x^+(t_0)$ is initialized by using the transient state probability at 10 seconds starting from an empty system.

Step 2: Find $x^-(t_i)$ and $\overline{x_\iota}$ using integration, as in equation (10).

Step 3: If $i < l$, then project $x^-(t_i)$ into the state space of period $[t_i, t_{i+1})$ to find $x^+(t_i)$ and go to step 2.

2.3.1.1 Projection Algorithm

The approach for projecting the probability vector $x^-(t_i)$ from a previous time period to vector $x^+(t_i)$ in subsequent time period as in step 3 is described in this section. Previous time period vectors $x^-(t_i)$ and subsequent time period vector $x^+(t_i)$ will be rewritten as $x^-$and x^- to simplify notation. The sets of the states of the inter-arrival distribution of the Coxian queue associated with the previous time period and subsequent time period will be defined as A^- and A^+, respectively, and the set of states of the customers in service in the Coxian queues will be defined as *S*. Let $x_{ij}^-(x_{ij}^+)$ be the associated component of vector $x^{-}(x^{+})$ for each state $i \in A^{-}$ or $i \in A^{+}$ and each state $j \in S$. The projection algorithm is described below.

Step 1: *Determine the probability state vector of the inter-arrival distribution* α^{-} *in the previous time period*. For each state, $i \in A^{-}$ set $\alpha_{i}^{-} = \sum_{j \in S} x_{ij}^{-}$.

Step 2: *Determine the probability state vector of the inter-arrival distribution* α^{+} *in the subsequent time period*. Solve the goal programming problem in (11-15) in which $0 < w_1 < w_2 < \cdots$,

$$min \sum_{i=1}^{\infty} (d_i + c_i) \tag{11}$$

$$\text{s.t.}\, i!\left(\alpha^{+}(T^{+})^{-i}1\right) - i!\left(\alpha^{-}(T^{-})^{-i}\right) + w_i d_i - w_i c_i = 0, \quad \forall i = 1, 2, \ldots \tag{12}$$

$$\sum_{i \in A^{+}} \alpha_i^{+} = 1 \tag{13}$$

$$\alpha_i^{+} \geq 0 \qquad \forall i \in A^{+} \tag{14}$$

$$c_i, d_i \geq 0 \qquad \forall i = 1, 2, \ldots \tag{15}$$

Objective (11) minimizes the difference between moments of the two arrival process distributions. The constraint set (12) is the goal constraint, which records the difference between moments of two distributions. Constraint (13) is the summation of the initial probability of arrival process, which must be 1, where α_i^{+} is the probability that the process starts at phase *i*. Constraint set (14) ensures each probability is nonnegative. In constraint set (15), c_i , d_i is the *i*th moment difference. For each *i*, if the moment of distribution associated with the subsequent time period is greater than the moment of the distribution associated with the previous time period, then $d_i = 0, c_i > 0$; otherwise $d_i \geq 0, c_i = 0$. The constant w_i is a penalty for difference in the i^{th} moment of the distributions associated with the consecutive time periods.

Step 3: *Determine the probability state vector* x^{+} *in the subsequent time period*. For each state $i \in A^{+}$ and each state $j \in S$ set $x_{ij}^{+} = \alpha_i^{+} \sum_{\bar{\imath} \in A^{-}} x_{\bar{\imath}j}^{-}$. Once an initial probability is determined for the next period, then continue to integrate and project for the following time period.

3 Validating Time Dependent Queuing Models with Cell-Level FACET Simulation Data

3.1 Results Comparing Time-Dependent Coxian and Markovian Queuing Models

The format of inter-arrival time and service time distributions determine the complexity of the queuing system. In practice, these distributions can take almost any format in real systems. In this section, two different time-dependent model results are compared. One, the inter-arrival time is a time-dependent exponential distribution, and the service time is also an exponential distribution, which forms an $M(t)/M/s/s$ queue. Two, the inter arrival time is a piecewise Coxian distribution, and the service time is a Coxian distribution as well, which forms an $C_{m(t)}(t)/C_k/s/s$ queue. As equation (4) shows, the state space of a Coxian queue increases very fast as the number of servers s and number of phases in the service time distribution increases. Thus, a trade off between accuracy and the number of phases needs to be made. Two- and three-phase Coxian distributions are widely used in previous research (Van Der Heijden, 1988; van der Laan and Salomon, 1997; Vidalis and Papadopoulos, 1999). In this research, the experiment shows that when a Coxian queue model exceeds 3 phases, the computational time becomes substantially longer. Thus for $C_{m(t)}(t)/C_k/s/s$ queues, the number of phases both inter-arrival and service time distributions are chosen to be either 2 or 3, depending upon which fits the data better. In the following section, the above two types of queuing models have been validated by the cell-level spatial resolution FACET data. Results of time invariant $M/M/s/s$ queues are given along with results of the aforementioned two models for comparison.

Results for models of five cells that are above the major airports ATL, DFW, JFK, LAX, and ORD are given in Figure 5 through Figure 9, which show the average number of aircraft in the system for each of the two time varying queuing models together with the time invariant queuing model. Capacity is a given constant value. The average number of aircraft in FACET simulation sample paths for June 1 to 7, 2007 is represented by the dotted lines, which were calculated by counting the number of aircraft at a particular cell every 30 seconds, and then averaging the total number of aircraft for every one-hour time period.

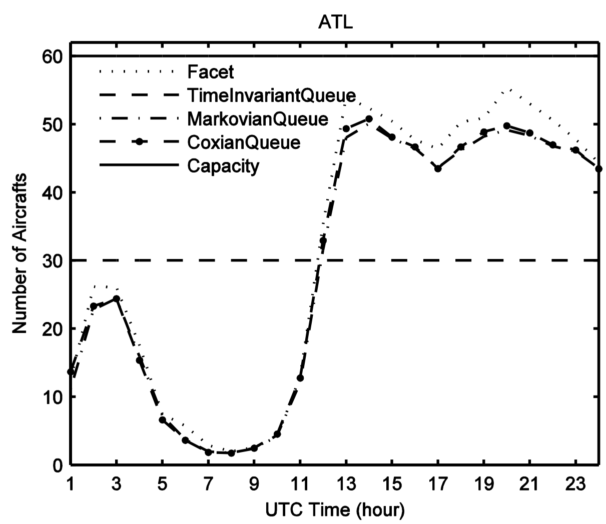


Figure 5. Number of aircraft in queuing models at ATL compared with FACET data

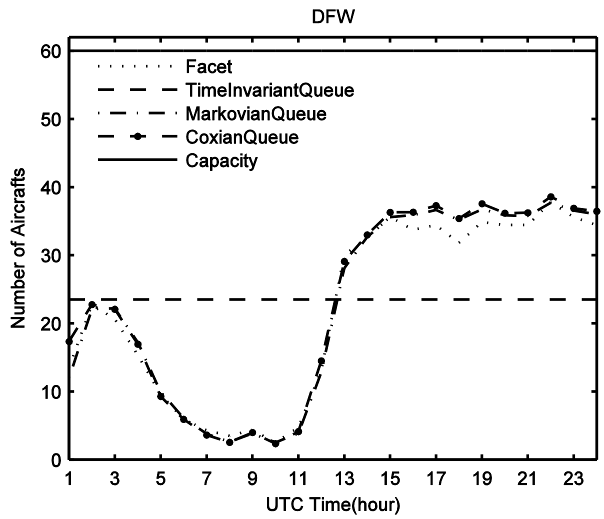


Figure 6. Number of aircraft in queuing models at DFW compared with FACET data

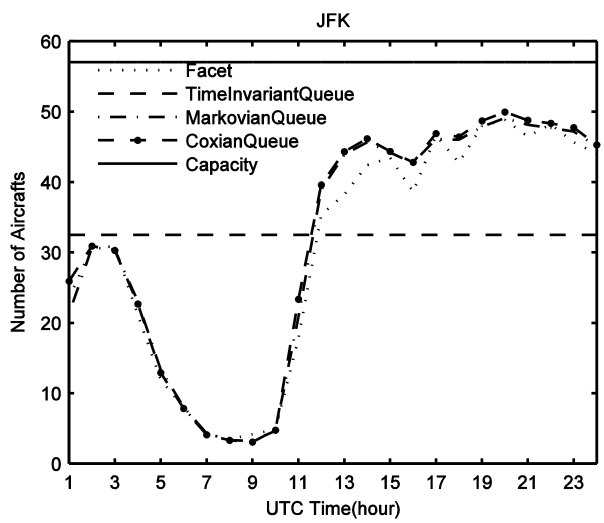


Figure 7. Number of aircraft in queuing models at JFK compared with FACET data

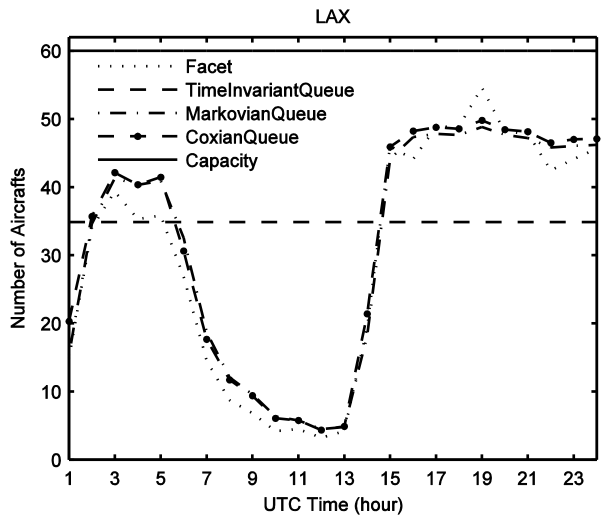


Figure 8. Number of aircraft in queuing models at LAX compared with FACET data

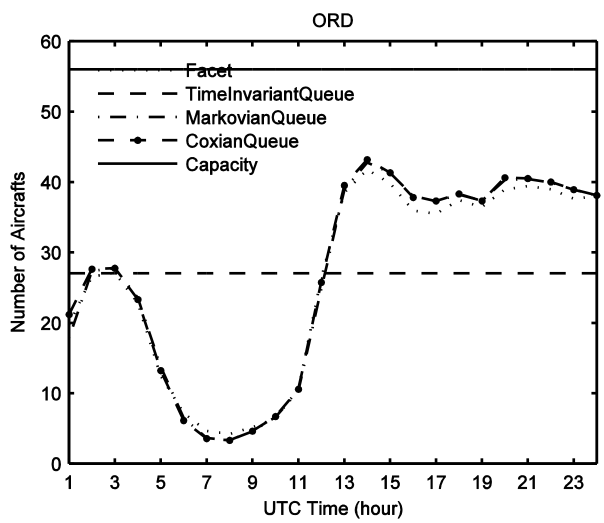


Figure 9. Number of aircraft in queuing models at ORD compared with FACET data

The results of the $M(t)/M/s/s$ model and the $C_{m(t)}(t)/C_k/s/s$ model capture the variation of demand very well in most time periods until time periods 12 or 13. Even though the Coxian distributions fit the arrival time and service time distributions more accurately, the $M(t)/M/s/s$ model and the $C_{m(t)}(t)/C_k/s/s$ models are extremely close at predicting the number of aircraft in the cell. Average percentage error of the $M(t)/M/s/s$ model and the $C_{m(t)}(t)/C_k/s/s$ model at ATL cell, DFW cell, JFK cell, LAX cell and ORD cell is 9.43% and 9.31%, 7.81% and 8.98%, 6.5% and 7.15%, 14.81% and 15.05%, 5.79% and 5.76% separately. However, the $M(t)/M/s/s$ models require only 7 to 8 seconds to solve, while the $C_{m(t)}(t)/C_k/s/s$ models require hours. Investigating the data, there is a sudden increase in the number of arriving aircraft compared to the previous period. This phenomena undermined the constant piecewise inter arrival assumption.

3.2 Comparison of Expected Transient Solution and Steady State Solution

In this section, the transient state approximation results are compared with steady state approximation results of queuing models $M(t)/M/s/s$ using cell level data. Percentage error is the difference between model results and FACET simulation results divided by the FACET simulation results then times 100%. Average percentage error for each cell is averaging percentage error of all 24 time periods. Comparison of the average percentage error between the transient solution and the steady state solution in cells is shown in Figure 10.

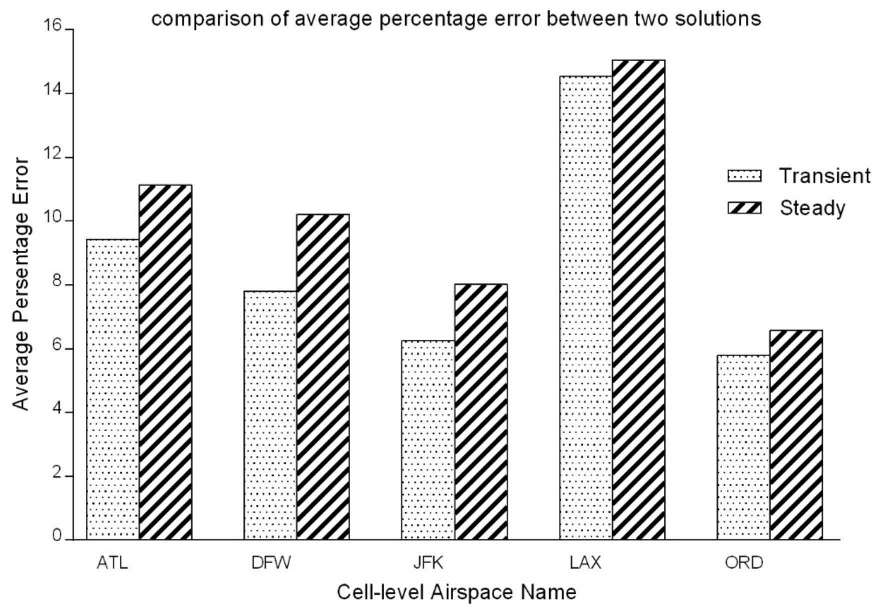


Figure 10. Comparison between transient and steady state solutions

Observe that the transient state approximation is more accurate than the steady state approximation results. Solutions from transient state approximation is 1.7%, 2.40%, 1.78%, 0.19%, 1.79% better than from steady state approximation for the five cells, respectively. Compared to steady state approximation, the accuracy of average percentage error from transient state solution has improved 15.27%, 23.51, 22.16%, 1.14%, 12.04% for the five cells.

3.3 Comparison of $M(t)/M/s/s$ and $M(t)/M(t)/s/s$ Models

In this section, we compare results for time-dependent and time-invariant service time distributions for the aforementioned five cells. In $M(t)/M(t)/s/s$ models, the time-dependent inter-arrival time distributions are approximated by piece-wise constant Markovian distributions each with 24 one-hour time periods, and the time-dependent service time distributions are approximated by piece-wise constant Markovian distributions each with 4 six-hour time periods. In the latter model, segmenting service time into a series of time periods could truncate the service time since certain flights make enter the cell in one period and leave in a subsequent period. Thus each individual time period must be large enough to avoid such truncated service time data. Here, we use 4 six-hour periods to approximate the service time distribution. Figure 11 is the average percentage error of both $M(t)/M/s/s$ and $M(t)/M(t)/s/s$ models.

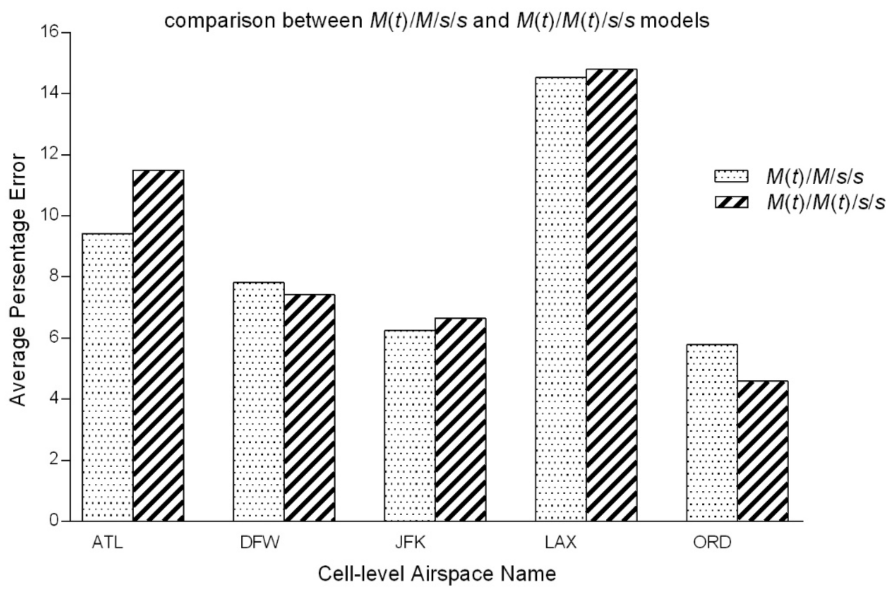


Figure 11. Comparison between $M(t)/M/s/s$ and $M(t)/M(t)/s/s$ models

The $M(t)/M(t)/s/s$ model is slightly more accurate over DFW and less than 2% more accurate over ORD, while the $M(t)/M/s/s$ is slightly more accurate over JFK and LAX and about 2% more accurate over ATL. Consequently, assuming that the service times follow time invariant distribution is reasonable.

4 Results and Discussion

Time-dependent queues with non-stationary arrival process are developed in this research. The $M(t)/M/s/s$ Markovian queuing models are built first. However, further statistical analysis shows inter-arrival and service times do not follow exponential distributions. With the advantage of closely approximating any arbitrary distribution without violating the Markovian property, Coxian distributions are employed to fit both arrival time and service time distribution. Thus, $C_{m(t)}(t)/C_k/s/s$ queues are developed in this research as well. Solving non-stationary queuing models is time consuming. The most commonly used method is approximating a non-stationary queue by stationary queues. In this research, an entire period is segmented into 24 one-hour flying periods. For each individual segment, a stationary arrival is assumed. However, each period is not independent. In every two consecutive periods, there exists a time epoch associated with two different probability state vectors. This research developed a linear programming projection algorithm that minimizes the difference between moments of these two distributions, which is a natural way to shift probability state vectors from one period to the next period.

Both $M(t)/M/s/s$ and $C_{m(t)}(t)/C_k/s/s$ queuing model validation results are given in Section 3. The validation results of cell-level data show both $M(t)/M/s/s$ and $C_{m(t)}(t)/C_k/s/s$ queuing models can accurately capture variation of the demand during each day. The method of segmenting the entire period into small periods and averaging over transient results of stationary queues in each time period successfully captures the variation of traffic demand. The validation results yield two conclusions. One, in each time period, $M(t)/M/s/s$ queuing models can approximate results as well as $C_{m(t)}(t)/C_k/s/s$ queues. However, calculating metrics of an $M(t)/M/s/s$ queue is substantially less computationally intensive than calculating those of an $C_{m(t)}(t)/C_k/s/s$ queue. In all five airspace cells, the computational times for $M(t)/M/s/s$ queues are between 7 to 8 seconds on a dual 2.2 GHz processor, while the computational times for $C_{m(t)}(t)/C_k/s/s$ queues are more than one hour. Consequently, this research shows that, in a pure loss model (waiting is not allowed), Markovian model results are as accurate as those of a Coxian queue. In addition, the validation results show that the average transient results for each one-hour segment are more accurate than steady state results. Finally, we compare $M(t)/M(t)/s/s$ models with non-stationary service time distributions with $M(t)/M/s/s$ models, and we find no improvement with the $M(t)/M(t)/s/s$ models.

The models in this research are described for cells over major airports. In particular, the $M(t)/M/s/s$ models are quick and accurate descriptions of these cells. Consequently, NASA can use them to analyze future NAS scenarios in which improvements to navigation and precision instrumentation has led to substantially more air traffic in the cells. Moreover, the $M(t)/M/s/s$, $C_{m(t)}(t)/C_k/s/s$, $M(t)/M(t)/s/s$ models are general, and the approaches developed in this research could be used for many applications with non-stationary time-dependent distributions. In future research, we will consider similar time-dependent queues for other air traffic models, such as runway models.

Acknowledgements

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