Comment on "Optimal Planner for Spacecraft Formations in Elliptical Orbits"

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Introduction

Zanon and Campbell[1] have recently presented a method of optimal control for spacecraft formation mission planning and reconfiguration, using Carter's solution[2] to the Tschauner-Hempel equations[3] as a basis. Reference 1 uses a spline function to approximate the forcing term appearing due to control in the relevant equations. The motivation behind the use of the spline function in this work is the lack of analytical solutions to certain key integrals. The purpose of this Comment is to show that the integrals in question can be solved for analytically.

Analysis

Reference 1 states in the paragraph succeeding Eq. (16) that "a considerable amount of difficulty arises when attempting to evaluate $Q(1)$ and $Q(4)$ because no closed-form solution has been found for the integration of $\rho^{-1}(\theta)\,K(\theta)$." The development in the succeeding sections in Ref. 1 then uses a spline function that approximates this integral, with varying degrees of accuracy, dependent on the eccentricity of the problem, and number of breakpoints in the spline function.

For practical applications, the spline function is shown to be a good approximation. This is shown in Fig. (1) of Ref. 1. The position error is as low as 1×10^{-4} m in some cases, and 1×10^{-1} m with 60 breakpoints and for an eccentricity of 0.95. However, the integral in question can be solved analytically. Let $I(\theta)$ denote this integral:

$$I(\theta) = \int \frac{1}{\rho(\theta)}\,K(\theta)\,d\theta \tag{1}$$

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where,

$$K(\theta) = \int \frac{\sin^2\theta}{\rho(\theta)^4}\,d\theta \tag{2}$$

$$\rho(\theta) = 1 + e\cos\theta \tag{3}$$

A change of variable of integration from true anomaly, θ, to eccentric anomaly, E, that was employed by Carter[2] to solve for $K(\theta)$, can also be used to solve for $I(\theta)$. The following relations are known:[4]

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E}, \quad \sin\theta = \frac{\eta\sin E}{1 - e\cos E} \tag{4a}$$

$$\cos E = \frac{\cos\theta + e}{1 + e\cos\theta}, \quad \sin E = \frac{\eta\sin\theta}{1 + e\cos\theta} \tag{4b}$$

where $\eta = \surd(1 - e^2)$. It follows that $d\theta = \eta\,dE/(1 - e\cos E)$. As shown in Ref. 2, the solution to $K(\theta)$ is:

$$K(\theta) = \int \frac{\sin^2\theta}{\rho(\theta)^4}\,d\theta = -\frac{1}{\eta^5}\int (1 - e\cos E)^2\,\sin^2 E\,dE \tag{5}$$

$$= \frac{1}{2\eta^5}\left(E - \frac{e}{2}\sin E - \frac{1}{2}\sin 2E + \frac{e}{6}\sin 3E\right) \tag{6}$$

In the case of the integral $I(\theta)$, the following relation is used:

$$\frac{d\theta}{\rho(\theta)} = \frac{d\theta}{1 + e\cos\theta} = \frac{1}{\eta}\,dE \tag{7}$$

Consequently,

$$I(\theta) = \int \frac{1}{\rho(\theta)}K(\theta)\,d\theta = \int \frac{1}{\eta}K\left(2\tan^{-1}\left\{\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right\}\right)\,dE \tag{8}$$

$$= \frac{1}{4\eta^6}\left(E^2 + e\cos E + \frac{1}{2}\cos 2E - \frac{e}{9}\cos 3E\right) \tag{9}$$

A similar integral that appears in Eqs. (11) and (14) of Ref. 1, given by $\rho(\theta)^{-2}\,K(\theta)$, can be solved for using the above steps, by noting that:

$$\frac{d\theta}{\rho(\theta)^2} = \frac{1}{\eta^3}(1 - e\cos E)\,dE \tag{10}$$

Even though the solution to $I(\theta)$ is comprised of terms involving E^2, the entire expression can be written in terms of $K(\theta)^2$, and harmonics of θ. Of these terms, $K(\theta)$ needs to be evaluated once in the entire procedure, and its value can be used to calculate $I(\theta)$ as well as the integral of $\rho(\theta)^{-2}\,K(\theta)$.

Conclusion

Since $Q(1)$ and $Q(4)$ can be solved for analytically, the authors' approach in Ref. 1 can be implemented using closed form solutions to these integrals, instead of spline approximations.

References

[1]Zanon, D. J. and Campbell, M. E., "Optimal Planner for Spacecraft Formations in Elliptical Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 1, January-February 2006, pp. 161–171.

[2]Carter, T. E., "State Transition Matrices for Terminal Rendezvous Studies: Brief Survey and New Examples," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 1, January-February 1998, pp. 148–155.

[3]Tschauner, J. F. A. and Hempel, P. R., "Rendezvous zu einemin Elliptischer Bahn umlaufenden Ziel," *Acta Astronautica*, Vol. 11, No. 2, 1965, pp. 104–109.

[4]Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, American Institute of Aeronautics and Astronautics, Inc., Reston, VA, revised ed., 1999, pp. 158–161.