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Elliptic Rendezvous in the Chaser Satellite Frame

Prasenjit Sengupta[1]

Abstract

The analysis of satellite rendezvous in planetary orbits typically derives control laws in a frame rotating with the target satellite. However, because the control law is ultimately required in the chaser satellite's frame, knowledge of the chaser satellite's motion with respect to the planet may be required to correctly transform the control laws. The transformation may also result in suboptimal or infeasible control laws if control components have different relative weights. A new approach is described that poses the rendezvous problem in the chaser satellite's frame directly. A nonlinear transformation between the chaser and target frames, in terms of relative position and velocity variables is derived. This transformation is used to formulate and solve the second-order nonlinear rendezvous problem using optimal power-limited propulsion analytically. Thus, a framework is developed that can be used to solve the orbital transfer problem. The efficacy of the derived control algorithm is demonstrated by means of an example.

Introduction

Satellite rendezvous and formation flight have been topics of great interest over the past decade, although historical interest in the problem has been present since the use of Hill's equations [1] for rendezvous near a circular orbit, by Clohessy and Wiltshire [2]. Analysis of both rendezvous and formation flight is similar in the sense that both require the understanding of the dynamics of relative motion, and derivation of control laws that can be used for docking, or formation establishment and reconfiguration.

Relative motion equations model the dynamics of one satellite (usually designated as the chaser or deputy), with respect to another satellite (designated as the target or chief), in a frame of reference affixed to the target satellite and rotating with satellite in its orbit. Studies on satellite rendezvous using optimal control typically penalize a quadratic cost function based on control effort [3–6]. However, the control variables in these equations are expressed in the frame rotating with the target satellite in its orbit. The control obtained from the analysis of these equations

[1]Research Scientist, Member, AAS, Optimal Synthesis Inc., 95 First Street, Suite 240, Los Altos, CA 94022. E-mail: sengupta@optisyn.com.

has to be transformed into the chaser's rotating frame to obtain appropriate values for physical application. To address this issue, this work poses the rendezvous problem using control variables expressed in the target's rotating frame directly. Although the resulting problem is a nonlinear optimal control problem, the framework enables the assessment of the effects of introducing relative frame transformations.

The article is organized as follows. The rendezvous equations are first presented, and expressions for finite rotations between chaser and target frames of reference are developed purely in terms of the relative position and velocity variables. These are then used to formulate new rendezvous equations with control effort expressed directly in the target's rotating frame. Perturbation methods are used to solve these equations including nonlinearities through the second order. By means of an example, the effects of using rotating frames on nonlinearity are presented.

Problem Formulation

Consider a rotating, Cartesian Local-Vertical Local-Horizontal (LVLH) frame of a satellite, with basis vectors $\{\mathbf{i_r}\ \mathbf{i_\theta}\ \mathbf{i_h}\}$. The vector $\mathbf{i_r}$ lies along the radius vector from the Earth's center to the satellite, $\mathbf{i_h}$ coinciding with the normal to the plane defined by the position and velocity vectors of the satellite, and $\mathbf{i_\theta} = \mathbf{i_h} \times \mathbf{i_r}$. In the rotating frame, the position and velocity of the target satellite are given by

$$\mathbf{r} = \{ r \quad 0 \quad 0 \}^\top \tag{1a}$$

$$\mathbf{v} = \{ v_r \quad v_\theta \quad 0 \}^\top \tag{1b}$$

The quantities r, v_r, and v_θ are obtained from the principles of two-body motion [7], as

$$r = \frac{p}{(1 + e\cos f)} \tag{2a}$$

$$v_r = \sqrt{\frac{\mu}{p}} e \sin f \tag{2b}$$

$$v_\theta = \sqrt{\frac{\mu}{p}} (1 + e\cos f) \tag{2c}$$

where $p = a(1 - e^2)$.

Let the relative position vector in the rotating frame be defined as $\varrho = \{ \xi \quad \vartheta \quad \zeta \}^\top$. The nonlinear relative motions can be obtained by twice differentiating the relative position in the rotating Cartesian LVLH frame, and equating the result to the sum of the differential gravity between the chaser and the target expressed in the LVLH frame, and external control input. The resulting nonlinear rendezvous equations are given by

$$\ddot{\xi} - 2\dot{\theta}\dot{\vartheta} - \dot{\theta}^2\xi - \ddot{\theta}\vartheta = -\frac{\mu(r + \xi)}{[(r + \xi)^2 + \vartheta^2 + \zeta^2]^{3/2}} + \frac{\mu}{r^2} + u_\xi \tag{3a}$$

$$\ddot{\vartheta} + 2\dot{\theta}\dot{\xi} - \dot{\theta}^2\vartheta - \ddot{\theta}\xi = -\frac{\mu\vartheta}{[(r+\xi)^2 + \vartheta^2 + \zeta^2]^{3/2}} + u_\vartheta \tag{3b}$$

$$\ddot{\zeta} = -\frac{\mu\zeta}{[(r+\xi)^2 + \vartheta^2 + \zeta^2]^{3/2}} + u_\zeta \tag{3c}$$

The equations for rendezvous can be written in normalized form, where the relative position is scaled by r. Furthermore, f is chosen as the independent variable, through the relation $\dot{f} = \upsilon_\theta / r$. These two steps result in position vector $\boldsymbol{\rho}$ and velocity vector $\boldsymbol{\rho}'$, which are given by

$$\boldsymbol{\rho} = \frac{1}{r}\varrho \tag{4a}$$

$$\boldsymbol{\rho}' = \frac{1}{\upsilon_\theta}\dot{\varrho} - \left(\frac{\upsilon_r}{\upsilon_\theta r}\right)\varrho \tag{4b}$$

Using the foregoing normalization, the nonlinear equations of motion can then be written as [8]:

$$x'' = 2y' + \frac{1+x}{(1+e\cos f)}\left(1 - \frac{1}{d^3}\right) + \frac{u_r}{(1+e\cos f)^3} \tag{5a}$$

$$y'' = -2x' + \frac{y}{(1+e\cos f)}\left(1 - \frac{1}{d^3}\right) + \frac{u_\theta}{(1+e\cos f)^3} \tag{5b}$$

$$z'' = -\frac{z}{(1+e\cos f)}\left(e\cos f + \frac{1}{d^3}\right) + \frac{u_h}{(1+e\cos f)^3} \tag{5c}$$

where $d = [(1+x)^2 + y^2 + z^2]^{1/2}$, and $\{u_r \quad u_\theta \quad u_h\}^\top = \{u_\xi \quad u_\vartheta \quad u_\zeta\}^\top / (\mu/p^2)$. Equation 5 can be rewritten as the nonlinear system

$$\mathbf{x}' = \mathbf{g}(f, \mathbf{x}) + B(f)u \tag{6}$$

where

$$\mathbf{x} = \{x \quad y \quad z \quad x' \quad y' \quad z'\}^\top \tag{7a}$$

$$\mathbf{u} = \{u_r \quad u_\theta \quad u_h\}^\top \tag{7b}$$

$$B(f) = \frac{1}{(1+e\cos f)^3}\begin{bmatrix} \mathbb{O}_3 \\ \mathbb{1}_3 \end{bmatrix} \tag{7c}$$

When the relative position is small compared with the radial distance of the target satellite from the gravitational center, that is, $\|\mathbf{x}\| \ll 1$, the nonlinear function $\mathbf{g}(f, \mathbf{x})$ can be expanded as

$$\mathbf{g}(f, \mathbf{x}) = A(f)\mathbf{x} + \tilde{\mathbf{g}}(f, \mathbf{x}) \tag{8}$$

where

$$A(f) = \begin{bmatrix} \mathbb{O}_3 & 1_3 \\ \tilde{A}(f) & \Omega \end{bmatrix},$$

$$\tilde{A}(f) = \begin{bmatrix} 3/(1 + e\cos f) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In equation 8, $\tilde{\boldsymbol{g}}(f, \boldsymbol{x})$ is composed of Legendre polynomials of second and higher order [5, 9, 10]. The linear form of the equations, where higher-order terms are neglected, are known as the Tschauner–Hempel (TH) equations [11], and have been used extensively to model the rendezvous problem with an eccentric reference or target orbit [4, 6, 12, 13]. The case $e = 0$, corresponding to a target in a circular orbit, results in the Clohessy–Wiltshire [2] model for relative motion. The circular case is not discussed here because a considerable amount of literature has been devoted to rendezvous near a circular orbit [14–16].

Although the linear form of equation 6 has been used extensively in the literature for control design, it does not address the fact that even if all the relative states are assumed observable, the control is expressed in the target's LVLH frame. Consequently, from the perspective of implementing a decentralized architecture, the control law may only be of limited use to the chaser because the chaser satellite requires the control law in its own LVLH frame.

To further explain this point, let the control vector in the chaser's LVLH frame be denoted by $\hat{\boldsymbol{u}}$, and C_{target} and C_{chaser} denote the direction cosine matrices that transform a vector in the Earth-centered inertial frame to the LVLH frames of the target and chaser, respectively. Then, the relative direction cosine matrix C_{rel}, that transforms a vector in the chaser's frame to the target's frame, is given by

$$C_{\text{rel}} = C_{\text{target}} C_{\text{chaser}}^{\top} \tag{9}$$

The control in the target's LVLH frame is then given by

$$\mathbf{u} = C_{\text{rel}} \hat{\mathbf{u}} \tag{10}$$

The relative direction cosine matrix C_{rel} can be obtained as a function of the relative states, in several ways. One method is to express the direction cosine matrices in equations 9 and 10 using suitable sets of parameters [17]. A natural choice is the 3–1-3 Euler angle set comprising Ω, i, and $\theta = \omega + f$, of the satellite's orbit, because these elements are part of the satellite's classical orbital element set. If it is assumed that the orbital elements of the chaser are not very different from those of the target, then the former can be written in terms of the latter, using Taylor series expansions. For example, one may obtain a set of differential orbital elements in terms of the relative states [18] as

$$\delta i \approx \sin\theta\, z + \cos\theta\, z' \tag{11a}$$

$$\delta\Omega \approx -\frac{\cos\theta}{\sin i} z + \frac{\sin\theta}{\sin i} z' \tag{11b}$$

$$\delta\theta \approx y + \cot i \cos\theta\, z - \cot i \sin\theta\, z' \tag{11c}$$

However, equations 11 can only be used to develop C_{rel} when the differences between the target and chaser's elements are small. Second-order transformations

[8] can be made use of, although their use is also limited by the fact that these are truncated series. Furthermore, no known formulae exist in the literature, for expansions to arbitrary order.

In reference [19], an approach was described in which relative LVLH states can be calculated from relative inertial states and the inertial states of the target satellite. Using key results from that work, the relative direction cosine matrix C_{rel} can be obtained in terms of the relative LVLH states. The full development of C_{rel} is included as an appendix. It can be shown that $C_{\text{rel}} \equiv C_{\text{rel}}(x)$; that is, the relative direction cosine matrix is purely a function of the normalized relative position and velocity. The use of equation 10 and equation 8 in equation 6 results in a system of equations of the form

$$\mathbf{x}' = A(f)\mathbf{x} + \tilde{\mathbf{g}}(f, \mathbf{x}) + \tilde{B}(f, \mathbf{x})\hat{\mathbf{u}} \tag{12}$$

where $\tilde{B}(f, \mathbf{x}) = B(f)\, C_{\text{rel}}(\mathbf{x})$.

The optimal control for minimum-fuel, power-limited rendezvous is solved by minimizing the cost function

$$\mathcal{J} = \frac{1}{2}\int_0^T \hat{\mathbf{u}}^\top R \hat{\mathbf{u}}\, \mathrm{d}t \tag{13}$$

where $R = \text{diag}(R_1, R_2, R_3) > 0$. The use of weights $R_{1,2,3}$ allows the inclusion of preferential firing algorithms because of unavailability of radial thrust [3], geometric or weight restrictions [20], or for the avoidance of plume impingement. The cost function for a control law in the target's frame, will in general be different from the cost function for a control in the chaser's frame, and consequently, the control law obtained from using a modified cost will also in general, be different. If the weights on the controls are identical, then $R_1 = R_2 = R_3 = R$, and

$$\mathcal{J} = \frac{R}{2}\int_0^T \hat{\mathbf{u}}^\top \hat{\mathbf{u}}\, \mathrm{d}t = \frac{R}{2}\int_0^T \mathbf{u}^\top C_{\text{rel}} C_{\text{rel}}^\top \mathbf{u}\, \mathrm{d}t = \frac{R}{2}\int_0^T \mathbf{u}^\top \mathbf{u}\, \mathrm{d}t \tag{14}$$

In fact, the fuel cost is the same irrespective of the frame in which the control is expressed, because

$$\text{fuel} \propto \int_0^T \mathbf{u}^\top \mathbf{u}\, \mathrm{d}t = \int_0^T \hat{\mathbf{u}}^\top \hat{\mathbf{u}}\, \mathrm{d}t \tag{15}$$

The optimal control for equation 12, is solved for the cost function

$$\mathcal{J} = \frac{1}{2}\int_{f_0}^{f_T} \frac{\hat{\mathbf{u}}^\top R \hat{\mathbf{u}}}{(1 + e\cos f)^2}\, \mathrm{d}f \tag{16}$$

where f_0 and f_T are the true anomaly of the target at epoch and final time T, and where a factor of $(1 + e \cos f)^2$ is introduced in the denominator of the cost function, when changing the independent variable from time (or mean anomaly) to true anomaly [3, 4, 6]. For ease of notation, let $\tilde{R}(f) = R/(1 + e\cos f)^2$.

Equations for the states, $x = \{x \quad y \quad z \quad x' \quad y' \quad z'\}^\top$, and costates, $\lambda = \{\lambda_x \quad \lambda_y \quad \lambda_z \quad \lambda_{x'} \quad \lambda_{y'} \quad \lambda_{z'}\}^\top$, can be obtained from the Hamiltonian of the system, and are given as [21]

$$\hat{\mathbf{u}} = -\tilde{R}^{-1}(f)\tilde{B}^\top(f, \mathbf{x})\lambda \tag{17a}$$

$$\mathbf{x}' = A(f)\mathbf{x} + \tilde{\mathbf{g}}(f, \mathbf{x}) - \tilde{B}(f, \mathbf{x})\tilde{R}^{-1}(f)\tilde{B}^\top(f, \mathbf{x})\lambda \tag{17b}$$

$$\lambda' = -A^\top(f)\lambda - \left[\frac{\partial}{\partial x}\tilde{\mathbf{g}}(f, \mathbf{x})\right]^\top \lambda + \lambda^\top \tilde{B}(f, \mathbf{x})\tilde{R}^{-1}(f)\left[\frac{\partial}{\partial x}\tilde{B}(f, \mathbf{x})\right]^\top \lambda \tag{17c}$$

$$\mathbf{x}(f_0) = \mathbf{x}_0, \qquad \mathbf{x}(f_T) = \mathbf{x}_T \tag{17d}$$

Equation 17 is a 12th-order system of nonlinear equations, with six initial and six final conditions on the states. It is worth noting that for satellite rendezvous, $\boldsymbol{x}_T = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^\top$, but the formulae derived in this article are valid for any value of desired final state vector. Therefore, the derivation is not limited to rendezvous, but can also be applied to formation reconfiguration and establishment. Furthermore, while the thrust-limited rendezvous problem is beyond the scope of the article, the analysis presented here still holds for the development of the costate equations that are used in conjunction with Prontryagin's Minimum Principle [21] to address the thrust-limited rendezvous problem.

Equation 17 can also be used to describe the nonlinear orbital transfer problem with power-limited propulsion, in a coordinate system rotating with the target. The optimal control can be determined if $\boldsymbol{\lambda}$ vector is known at all instants of time, and the evolution of $\boldsymbol{\lambda}$ with time requires that the initial value $\boldsymbol{\lambda}_0$ be determined. The initial costate vectors can be obtained by numerically iterative procedures such as shooting methods; however, depending on the length of the simulation, these methods can require large computation times. Furthermore, these methods require accurate initial guesses. Recent work on the optimal control of nonlinear systems uses generating functions [15] to obtain $\boldsymbol{\lambda}_0$. The formulation used in this article is in the same category as the Tschauner–Hempel equations, in that the system is nonautonomous. The additional complexity involved with analyzing nonautonomous systems cannot be avoided unless the target orbit eccentricity is assumed zero.

Optimal Control Problem in the Chaser's Frame

In this section, the system specified by equation 17 is simplified by utilizing the fact that the relative motion is small compared with the size of the target's orbit. This approach also enables a comparison of the relative magnitudes of the nonlinear terms. As shown in the appendix, the relative direction cosine matrix can be written as a series of matrices whose entries are polynomials in the states. To the first order

$$C_{\text{rel}} \approx \begin{bmatrix} 1 & -y & -z \\ y & 1 & -z' \\ z & z' & 1 \end{bmatrix} = \mathbb{1}_3 + D \times (\mathbf{x}) \tag{18}$$

where $\boldsymbol{D} \times (\mathbf{x})$ is a skew-symmetric matrix.

Upon substituting equation 18 in equation 17, a set of approximate equations for the states and costates are obtained as

$$\begin{aligned}\mathbf{x}' \approx\ & A(f)\mathbf{x} - B(f)\tilde{R}^{-1}(f)B^\top(f)\lambda \\ & + \tilde{\mathbf{g}}(f, \mathbf{x}) - B(f)[D \times (\mathbf{x})\tilde{R}^{-1}(f) - \tilde{R}^{-1}(f)D \times (\mathbf{x})]\lambda\end{aligned} \tag{19a}$$

$$\lambda' \approx -A^\top(f)\lambda - \left[\frac{\partial}{\partial \mathbf{x}}\tilde{\mathbf{g}}(\mathbf{x})\right]^\top \lambda - \lambda^\top B(f)\left[\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}D \times (\mathbf{x})\right]\tilde{R}^{-1}(f)B^\top(f)\lambda \tag{19b}$$

In equation 19b, $D = \mathrm{d}D \times (\mathbf{x})/\mathrm{d}x$ is a third-order tensor of dimension $6 \times 3 \times 3$, whose nonzero entries are given by

$$D_{212} = D_{313} = D_{623} = -1, \quad D_{221} = D_{331} = D_{632} = 1 \tag{20}$$

Equation 19 is a system of ordinary differential equations in states and costates, with second-order nonlinearities introduced by the relative frame between the chaser and target's respective LVLH frames. The introduction of second-or higher-order terms in equation 11 would result in terms through the third order in equation 19. Therefore, the linear approximation in equation 11 is considered valid.

As shown in reference [10], the nonlinear function $\tilde{\mathbf{g}}(f, \mathbf{x})$ represents nonlinearities in the differential gravity field of second and higher order. This function may be written as

$$\tilde{\mathbf{g}}(f, \mathbf{x}) = \sum_{k=3}^{\infty} \frac{(-1)^k k}{(1 + e\cos f)(y^2 + z^2)}\left[\rho^k P_k(x/\rho)\begin{Bmatrix}0\\0\\0\\0\\y\\z\end{Bmatrix} + \rho^{k-1}P_{k-1}(x/\rho)\begin{Bmatrix}0\\0\\0\\y^2+z^2\\-xy\\-xz\end{Bmatrix}\right] \tag{21}$$

where $\rho = \|\boldsymbol{\rho}\|$.

Because the approximation of C_{rel} introduces nonlinearities of the second order, the function $\tilde{\mathbf{g}}(f, \mathbf{x})$ is also restricted to the second order

$$\tilde{\mathbf{g}}(f, \mathbf{x}) = \frac{3}{2}\frac{1}{(1 + e\cos f)}\{0 \quad 0 \quad 0 \quad (y^2 + z^2 - 2x^2) \quad 2xy \quad 2xz\}^\top \tag{22}$$

The implication of the introduction of second-order terms in the equations is that whenever relative distance cannot be assumed small in comparison with the distance of the target from the planet, the relative frame kinematics must also be taken into account in the formulation.

Perturbation Approach to the Solution of the Optimal Control Problem

A solution based on perturbation techniques can now be developed for equation 19, with the inclusion of quadratic terms because of differential gravity, given by equation 21. The state and costate equations are rewritten as

$$\begin{Bmatrix}\mathbf{x}'\\ \lambda'\end{Bmatrix} = \begin{bmatrix}A(f) & -B(f)\tilde{R}^{-1}(f)B^\top(f)\\ \mathbb{O}_6 & -A^\top(f)\end{bmatrix}\begin{Bmatrix}\mathbf{x}\\ \lambda\end{Bmatrix} + \mathbf{d}(f, \mathbf{x}, \boldsymbol{\lambda}) \tag{23}$$

where $\mathbf{d}(f, \mathbf{x}, \lambda)$ is the vector composed of terms of second order in $\mathbf{x}$ and λ, given by

$$\mathbf{d}(f, \mathbf{x}, \lambda) = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \dfrac{3(z^2 + y^2 - 2x^2)}{2(1 + e\cos f)} + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\dfrac{y\lambda_{y'}}{(1 + e\cos f)^4} - \left(\dfrac{1}{R_1} - \dfrac{1}{R_3}\right)\dfrac{z\lambda_{z'}}{(1 + e\cos f)^4} \\ \dfrac{3xy}{(1 + e\cos f)} + \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\dfrac{y\lambda_{x'}}{(1 + e\cos f)^4} + \left(\dfrac{1}{R_3} - \dfrac{1}{R_2}\right)\dfrac{z'\lambda_{z'}}{(1 + e\cos f)^4} \\ \dfrac{3xz}{(1 + e\cos f)} + \left(\dfrac{1}{R_3} - \dfrac{1}{R_2}\right)\dfrac{z'\lambda_{y'}}{(1 + e\cos f)^4} - \left(\dfrac{1}{R_1} - \dfrac{1}{R_3}\right)\dfrac{z\lambda_{x'}}{(1 + e\cos f)^4} \\ -\dfrac{3(y\lambda_{y'} + z\lambda_{z'} - 2x\lambda_{x'})}{(1 + e\cos f)} \\ -\dfrac{3(x\lambda_{y'} + y\lambda_{x'})}{(1 + e\cos f)} - \left(\dfrac{1}{R_2} - \dfrac{1}{R_1}\right)\dfrac{\lambda_{x'}\lambda_{y'}}{(1 + e\cos f)^4} \\ -\dfrac{3(x\lambda_{z'} + z\lambda_{x'})}{(1 + e\cos f)} + \left(\dfrac{1}{R_1} - \dfrac{1}{R_3}\right)\dfrac{\lambda_{x'}\lambda_{z'}}{(1 + e\cos f)^4} \\ 0 \\ 0 \\ -\left(\dfrac{1}{R_3} - \dfrac{1}{R_2}\right)\dfrac{\lambda_{y'}\lambda_{z'}}{(1 + e\cos f)^4} \end{array}\right\} \tag{24}$$

Reference [5] shows that the kth-order nonlinear polynomial in the expansion of the differential gravity is $\mathbb{O}(\rho_0^k/\rho^k)$ where ρ_0 represents the relative orbit size, and can be equal to the initial relative distance between the target and chaser. Consequently, the second-order nonlinearity is approximately (ρ_0/p) times smaller than the linear term, and it is therefore assumed that the solution to equation 23 can be written as a perturbation to a given reference solution [22]

$$\mathbf{x} \approx \mathbf{x}_{\text{ref}} + \tilde{\mathbf{x}} \tag{25a}$$

$$\lambda \approx \lambda_{\text{ref}} + \tilde{\lambda} \tag{25b}$$

where $\mathbf{x}_{\text{ref}}$ and λ_{ref} are the solutions to the linear problem. An analytical solution to the linear problem has been derived in reference [6], and is of the form

$$\begin{Bmatrix} \mathbf{x}_{\text{ref}}(f) \\ \lambda_{\text{ref}}(f) \end{Bmatrix} = \Phi(f, f_0) \begin{Bmatrix} \mathbf{x}_{\text{ref}}(f_0) \\ \lambda_{\text{ref}}(f_0) \end{Bmatrix} \tag{26}$$

$$\begin{aligned} \Phi(f, f_0) &= \begin{bmatrix} \Phi_{xx}(f, f_0) & \Phi_{x\lambda}(f, f_0) \\ \mathbb{O}_6 & \Phi_{\lambda\lambda}(f, f_0) \end{bmatrix} \\ &= \begin{bmatrix} L(f)M(f_0) & -L(f)[N(f) - N(f_0)]L^\top(f_0) \\ \mathbb{O}_6 & M^\top(f)L^\top(f_0) \end{bmatrix} \end{aligned} \tag{27}$$

The matrices $L(f)$, $M(f)$, and $N(f)$ are available as appendices in reference [6] and are not reproduced here for the sake of brevity.

Upon substituting equation 25 in equation 23, and isolating the terms of second order, a set of equations are obtained as

$$\begin{Bmatrix} \tilde{\mathbf{x}}' \\ \tilde{\lambda}' \end{Bmatrix} = \begin{bmatrix} A(f) & -B(f)\,\tilde{R}^{-1}(f)B^{\top}(f) \\ \mathbb{O}_6 & -A^{\top}(f) \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{x}} \\ \tilde{\lambda} \end{Bmatrix} + \mathbf{d}(f, \mathbf{x}_{\mathrm{ref}}(f), \lambda_{\mathrm{ref}}(f)) \tag{28}$$

The solution to equation 28 is then given by

$$\begin{Bmatrix} \tilde{\mathbf{x}}(f) \\ \tilde{\lambda}(f) \end{Bmatrix} = \Phi(f, f_0) \begin{Bmatrix} \tilde{\mathbf{x}}(f_0) \\ \tilde{\lambda}(f_0) \end{Bmatrix} + \int_{f_0}^{f} \Phi(f, s)\mathbf{d}(s, \mathbf{x}_{\mathrm{ref}}(s), \lambda_{\mathrm{ref}}(s))\mathrm{d}s \tag{29}$$

However, from the initial and final conditions specified in equation 17,

$$\mathbf{x}(f_0) = \mathbf{x}_0 \Rightarrow \tilde{\mathbf{x}}(f_0) = 0 \tag{30a}$$

$$\mathbf{x}(f_T) = \mathbf{x}_T \Rightarrow \tilde{\mathbf{x}}(f_T) = 0 \tag{30b}$$

It follows that the initial value of $\boldsymbol{\lambda}$, that can be used to solve the optimal control problem, is given by

$$\begin{aligned} \lambda(f_0) = {} & \Phi_{x\lambda}^{-1}(f_T, f_0)\{\mathbf{x}_T - \Phi_{xx}(f_T, f_0)\mathbf{x}_0\} \\ & - \Phi_{x\lambda}^{-1}(f_T, f_0)\int_{f_0}^{f_T} [\Phi_{xx}(f_T, s)\Phi_{x\lambda}(f_T, s)]\mathbf{d}(s, \mathbf{x}_{\mathrm{ref}}(s), \lambda_{\mathrm{ref}}(s))\mathrm{d}s \end{aligned} \tag{31}$$

In terms of the known matrices $L(f)$, $M(f)$, and $N(f)$, the foregoing equation can be rewritten as

$$\begin{aligned} \lambda(f_0) = {} & -M^{\top}(f_0)[N(f_T) - N(f_0)]^{-1}\{[M(f_T)\mathbf{x}_T - M(f_0)\mathbf{x}_0] - \int_{f_0}^{f_T} M(s)\mathbf{d}_x(s)\mathrm{d}s \\ & + \int_{f_0}^{f_T} [N(f_T) - N(s)]L^{\top}(s)\mathbf{d}_{\lambda}(s)\mathrm{d}s\} \end{aligned} \tag{32}$$

where $\mathbf{d}_x(s)$ and $\mathbf{d}_\lambda(s)$ are the first six and last six elements of the vector $\mathbf{d}(s, \mathbf{x}_{\mathrm{ref}}(s))$, respectively. Although an analytical solution to the quadratures in equation 32 is not known, they can be evaluated easily using numerical integration or by spline approximations. Once equation 32 is evaluated, a feedback control from relative states x at true anomaly f, to rendezvous at states $\mathbf{x}_T$, at true anomaly f_T, can be obtained by substituting $\boldsymbol{\lambda}(f_0)$ into equation 29 and then calculating the optimal control using equation 17a. In terms of the matrices $L(f)$, $M(f)$, and $N(f)$, the feedback control law can be written as

$$\begin{aligned} \hat{\mathbf{u}}(\mathbf{x}, \mathbf{x}_T, f, f_T) = {} & \tilde{R}^{-1}(f)C_{\mathrm{rel}}(x)^{\top}\tilde{B}^{\top}(f)M^{\top}(f)[N(f_T) - N(f)]^{-1} \\ & \times \left\{ [M(f_T)\mathbf{x}_T - M(f)\mathbf{x}] - \int_{f}^{f_T} M(s)\mathbf{d}_x(s)\mathrm{d}s + \int_{f}^{f_T} [N(f_T) - N(s)]L^{\top}(s)\mathbf{d}_{\lambda}(s)\mathrm{d}s \right\} \end{aligned} \tag{33}$$

Numerical Simulations

The use of the control law presented in equation 33 is demonstrated on an example rendezvous mission, where the chaser satellite is initially placed in a periodic relative orbit around the target. The orbital elements of the target satellite are chosen as

$$a = 8285.17\text{ km}, \quad e = 0.2, \quad i = 50^\circ, \quad \Omega = 36^\circ, \quad \omega = 24^\circ, \quad M_0 = 4^\circ \tag{34}$$

The chaser satellite has the following initial states, chosen arbitrarily, but constrained to result in a periodic and bounded relative orbit in the absence of control

$$\begin{aligned} \xi_0 &= 49.51\text{ km}, \quad & \vartheta_0 &= 63.97\text{ km}, \quad & \zeta_0 &= 36.70\text{ km} \\ \dot{\xi}_0 &= 0.044\text{ km/s}, \quad & \dot{\vartheta}_0 &= -0.116\text{ km/s}, \quad & \dot{\zeta}_0 &= 0.013\text{ km/s} \end{aligned} \tag{35}$$

Upon scaling the relative position variables and relative velocity variables using equation 4, the initial state vector is given by

$$\mathbf{x}_0 = \{7.46 \quad 9.64 \quad 5.53 \quad 5.00 \quad -13.81 \quad 1.41\}^\top \times 10^{-3}, \qquad f_0 = 0.1068\text{rad} \tag{36}$$

Rendezvous with the target is given by the conditions

$$\mathbf{x}_T = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^\top, \qquad f_T = 3.6257\text{rad} \tag{37}$$

The value of f_T chosen in the foregoing equation corresponds to a mean anomaly of the target in its orbit given by $M_T = 220^\circ$, or approximately 75 min from epoch.

Rendezvous is desired without the use of radial thrust in the chaser's own LVLH frame. To obtain such a control law, the weights on the control are chosen as $R_1 = 1 \times 10^6$, $R_2 = 1$, $R_3 = 1$.

Semi-Analytical Solution for the Two-Point Boundary Value Problem

As shown in previous sections, the evaluation of $\boldsymbol{\lambda}_0$ allows the calculation of $\boldsymbol{\lambda}$ and consequently, the optimal control. A two-point boundary value problem for the second order system, given by equation 23, is solved using numerical shooting methods, for $\mathbf{x}_0$ and $\mathbf{x}_T$ given by equations 36 and 37, respectively.

Using the linear part of equation 32 results in $\lambda_{0_{\text{linear}}}$ that is obtained analytically as

$$\lambda_{0_{\text{linear}}} = \{70.45 \quad -14.86 \quad 2.89 \quad 36.91 \quad 25.97 \quad 0.00\}^\top \times 10^{-3} \tag{38}$$

This solution can be improved upon by adding the second order term in equation 32, upon evaluating the integral numerically. This results in $\lambda_{0_{\text{nonlinear}}}$, given as

$$\lambda_{0_{\text{nonlinear}}} = \{71.01 \quad -14.83 \quad 3.14 \quad 37.33 \quad 26.45 \quad 0.23\}^\top \times 10^{-3} \tag{39}$$

The true costate vector is found by numerically solving the two-point boundary value problem for the nonlinear system, using $\lambda_{0_{\text{linear}}}$ as an initial guess, and is given by

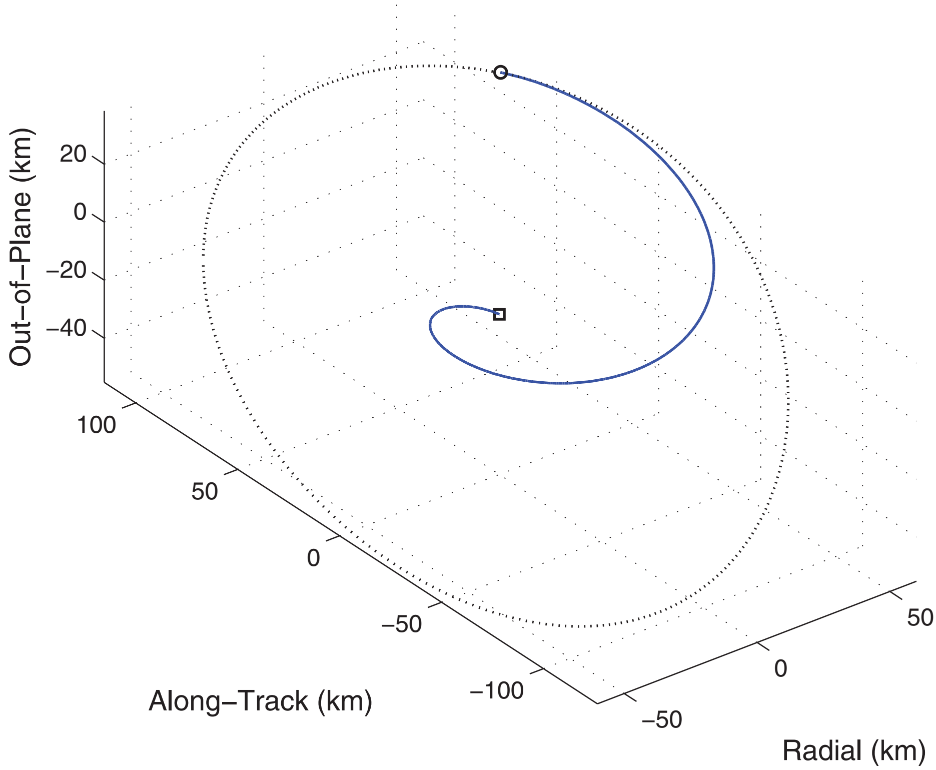

FIG. 1. Relative Trajectory.

$$\lambda_0 = \{\, 71.03 \quad -14.83 \quad 3.15 \quad 37.34 \quad 26.47 \quad 0.24 \,\}^{\top} \times 10^{-3} \tag{40}$$

This demonstrates that by using closed-form solutions and evaluating quadratures, equation 32 is an excellent approximation to the solution of the nonlinear rendezvous problem posed in this article.

Rendezvous Using Feedback Control

The control law given by equation 33 is now used to perform rendezvous given by conditions equations 36 and 37. Although the control is obtained by an approximate solution to approximate equations, it is applied to the nonlinear system given by equation 5. Figure 1 depicts the relative trajectory (solid line) starting from initial condition $\boldsymbol{x}_0$ at f_0, to rendezvous state $\boldsymbol{x}_T$ at f_T. The initial and final positions are marked by a circle and square, respectively, and the dotted trajectory represents the bounded relative orbit corresponding to initial conditions $\boldsymbol{x}_0$.

It should be noted that the second order form of the rendezvous equations, given by equation 23, can be written without formulating the control in the chaser frame, by ignoring terms of the second order dependent on $1/R_1$, $1/R_2$, and $1/R_3$, in equation 24. Therefore, two control laws are obtained: one that minimizes the cost function $(1/2)\int_{f_0}^{f_T} \mathbf{u}^{\top} \tilde{R} \mathbf{u} \, df = (1/2)\int_{f_0}^{f_T} \hat{\mathbf{u}}^{\top} C_{\text{rel}}^{\top} \tilde{R} C_{\text{rel}} \hat{\mathbf{u}} \, df$, and one that minimizes $(1/2)\int_{f_0}^{f_T} \hat{\mathbf{u}}^{\top} \tilde{R} \hat{\mathbf{u}} \, df$. The first control law is denoted by $\hat{\mathbf{u}}_1$, and the second control is denoted by $\hat{\mathbf{u}}_2$, where $\hat{\mathbf{u}}_1$ ignores the effects of the frame differences between chaser and target satellites. The control history using the two control laws is shown in Fig. 2.

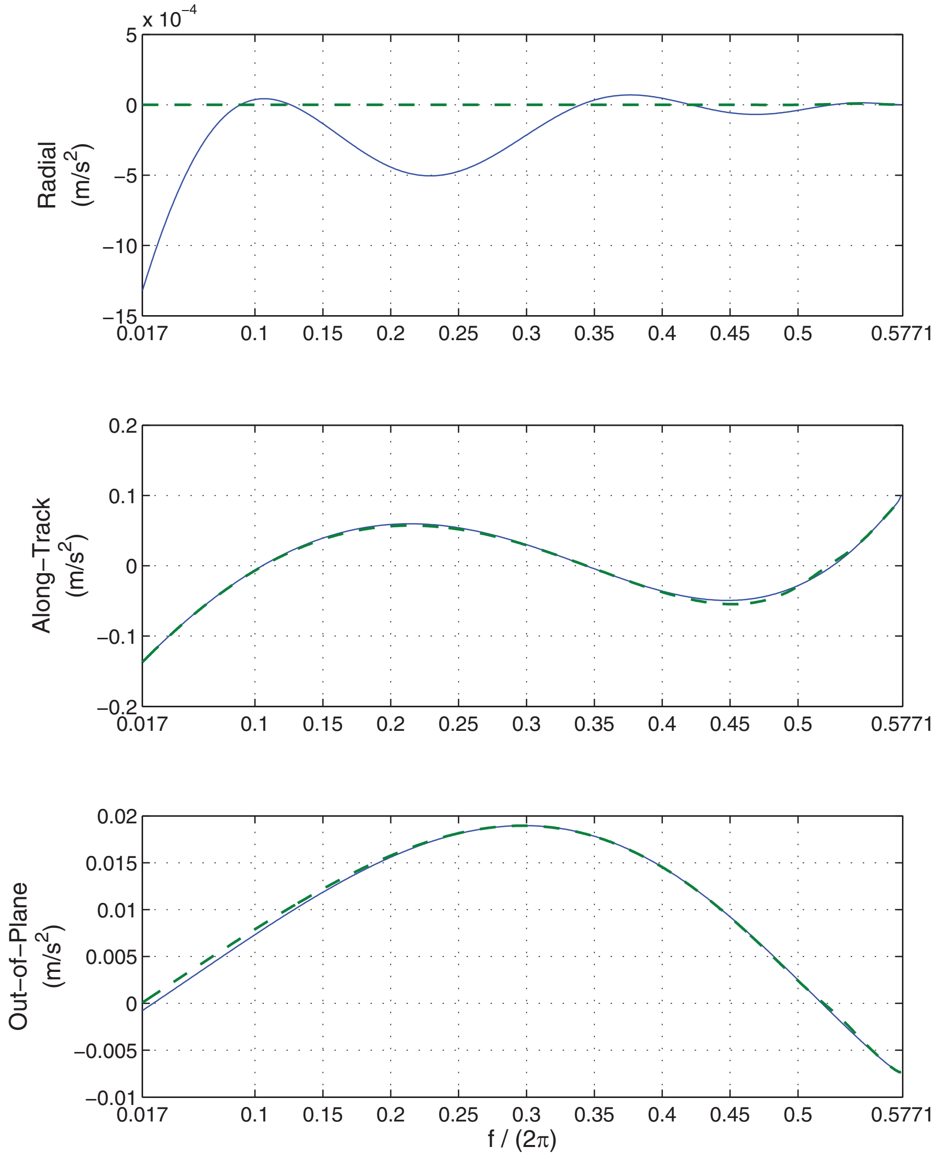

FIG. 2. Control History in the Frame of the Chaser Satellite, Minimization in Target Frame (Solid), and Minimization in Chaser Frame (Broken).

The solid line depicts the control history if the matrix C_{rel} is ignored in the control formulation, resulting in control law $\hat{\mathbf{u}}_1$, that is obtained by transforming the control law from the target LVLH frame. The broken line depicts the control law obtained by optimizing the cost function accounting for relative frame differences. The radial control in the chaser frame from $\hat{\mathbf{u}}_2$ is near-zero, whereas the radial control by using $\hat{\mathbf{u}}_1$ is not correctly minimized. It is also observed that as the chaser satellite approaches the target in its orbit, the difference between the two control laws is reduced, because of the decrease in nonlinearity introduced by frame difference. The magnitude of the control in the along-track and out-of-plane are

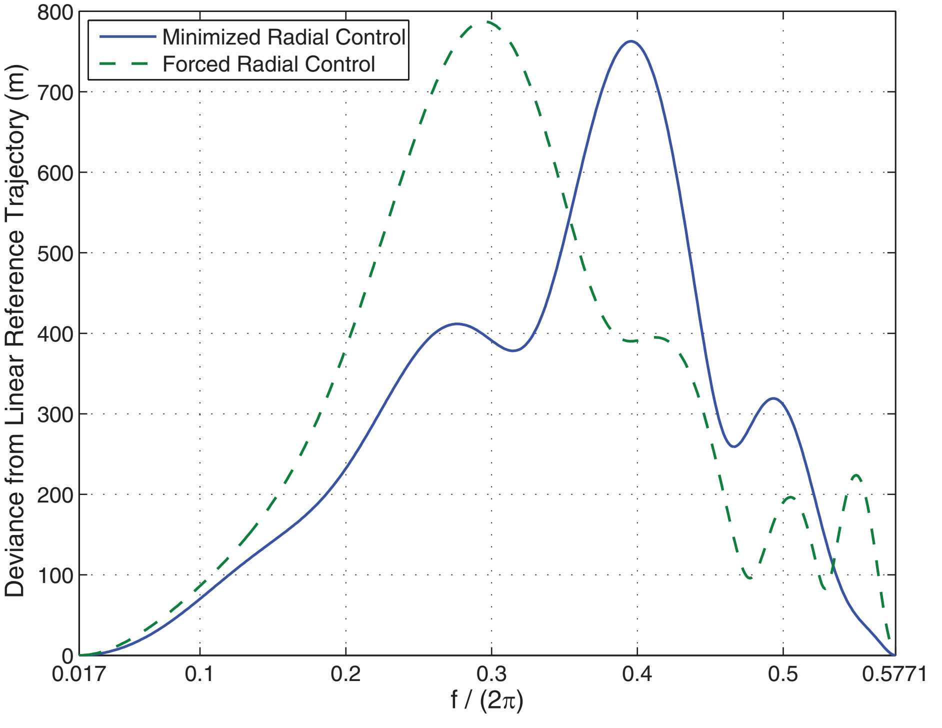

FIG. 3. Relative Trajectory Deviation From Linear Reference.

comparable; the difference in their respective values arises due the fact that in the new control law presented in this article, they have to compensate for a lack of availability of control in the radial direction.

To analyze the difference between control laws resulting from the minimization of different cost functions, the radial component of both control laws in their respective chaser frames are set to zero, translated back into the target frame, and applied to equation 5. Although the difference in resulting relative position can be several hundred meters, it is too small to depict on the scale of Fig. 1. Therefore, these trajectories are shown as deviations from a reference trajectory obtained by applying a linear control law (i.e., without any second-order terms), in Fig. 3. In this figure, the solid line depicts the trajectory obtained by using $\hat{\mathbf{u}}_1$, after forcing $\hat{u}_{1r} = 0$. the broken line depicts the trajectory obtained by using $\hat{\mathbf{u}}_2$, where the radial control is very near zero; in fact, forcing $\hat{u}_{2r} = 0$ causes insignificant change in the trajectory. The application of both nonlinear feedback control laws causes deviation from the linear reference trajectory, and the deviation can be as large as 800 m. Furthermore, a difference is noted in the relative trajectories obtained using control laws where radial thrust has been penalized, and where radial thrust has been forced to zero. Although the along track displacement is as large as 70 km in this example, the noted deviation from trajectory is large enough to have implications on the guidance law used in rendezvous. It is worth noting that even though the radial control is not used, the problem is still controllable in the linear sense [3]. However, the along-track control history will differ, to compensate for the lack of available radial control.

Conclusions

In this article, the problem of rendezvous has been analyzed, with control law derivation directly in the frame of the chaser satellite. A formulation for the relative direction cosine matrix between chaser and target frames, in terms of relative states, has been derived, and it has been shown that using this matrix introduces a state-dependent control influence matrix in the rendezvous equations.

From a general formulation for the nonlinearities in the rendezvous equations, polynomial series have been constructed, which allows for the use of perturbation theory as a solution methodology for optimal orbital transfer. A feedback control law has been derived that accounts for second-order nonlinearities in the rendezvous equations. It has been shown that introduction of relative frame effects is particularly useful when some components of the control thrust are penalized to avoid their use. This nonlinear feedback law is also useful for rendezvous missions where the relative distances are large. Furthermore, the results obtained are valid for elliptic orbits with no limitation on eccentricity. While in some cases, the control law obtained using the modified formulation is not very different from the conventional approach, the resulting trajectories can be different. The formulation developed in the article also serves as a framework for the orbital transfer problem to obtain the optimal control in a frame rotating with the target satellite in its orbit.

Appendix: Development of the Relative Direction Cosine Matrix

The relative direction cosine matrix C_{rel} is obtained in terms of the states $\{x \quad y \quad z \quad x' \quad y' \quad z'\}^\top$. The direction cosine matrix C_{target} can be obtained from the inertial radius and velocity vectors, $\mathbf{r}$ and $\mathbf{v}$, respectively. By using the angular momentum vector $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, and the vector $\mathbf{c}$ orthogonal to $\mathbf{r}$ and $\mathbf{h}$, such that $\mathbf{c} = \mathbf{h} \times \mathbf{r}$, the matrix C_{target} is given by [19]

$$C_{\mathrm{target}} = \begin{bmatrix} \dfrac{\mathbf{r}}{|\mathbf{r}|} & \dfrac{\mathbf{c}}{|\mathbf{c}|} & \dfrac{\mathbf{h}}{|\mathbf{h}|} \end{bmatrix}^\top \tag{41}$$

Similarly, the direction cosine matrix for the chaser's LVLH frame can be obtained from the inertial position, $\mathbf{r} + \boldsymbol{\delta}\mathbf{r}$, and velocity, $\mathbf{v} + \boldsymbol{\delta}\mathbf{v}$, of the chaser satellite in the inertial frame, where $\boldsymbol{\delta}\mathbf{r}$ and $\boldsymbol{\delta}\mathbf{v}$ are the relative position and velocity, respectively, of the chaser with respect to the target in the inertial frame.

$$C_{\mathrm{chaser}} = \begin{bmatrix} \dfrac{\mathbf{r} + \delta\mathbf{r}}{|\mathbf{r} + \delta\mathbf{r}|} & \dfrac{\mathbf{c} + \delta\mathbf{c}}{|\mathbf{c} + \delta\mathbf{c}|} & \dfrac{\mathbf{h} + \delta\mathbf{h}}{|\mathbf{h} + \delta\mathbf{h}|} \end{bmatrix}^\top \tag{42}$$

It follows that the relative direction cosine matrix can be written as

$$C_{\mathrm{rel}} = \begin{bmatrix} \dfrac{\mathbf{r}}{|\mathbf{r}|} & \dfrac{\mathbf{c}}{|\mathbf{c}|} & \dfrac{\mathbf{h}}{|\mathbf{h}|} \end{bmatrix}^\top \begin{bmatrix} \dfrac{\mathbf{r} + \delta\mathbf{r}}{|\mathbf{r} + \delta\mathbf{r}|} & \dfrac{\mathbf{c} + \delta\mathbf{c}}{|\mathbf{c} + \delta\mathbf{c}|} & \dfrac{\mathbf{h} + \delta\mathbf{h}}{|\mathbf{h} + \delta\mathbf{h}|} \end{bmatrix} \tag{43}$$

$$= \begin{bmatrix} \dfrac{\mathbf{r}^\top(\mathbf{r} + \delta\mathbf{r})}{|\mathbf{r}||\mathbf{r} + \delta\mathbf{r}|} & \dfrac{\mathbf{r}^\top(\mathbf{c} + \delta\mathbf{c})}{|\mathbf{r}||\mathbf{c} + \delta\mathbf{c}|} & \dfrac{\mathbf{r}^\top(\mathbf{h} + \delta\mathbf{h})}{|\mathbf{r}||\mathbf{h} + \delta\mathbf{h}|} \\ \dfrac{\mathbf{c}^\top(\mathbf{r} + \delta\mathbf{r})}{|\mathbf{c}||\mathbf{r} + \delta\mathbf{r}|} & \dfrac{\mathbf{c}^\top(\mathbf{c} + \delta\mathbf{c})}{|\mathbf{c}||\mathbf{c} + \delta\mathbf{c}|} & \dfrac{\mathbf{c}^\top(\mathbf{h} + \delta\mathbf{h})}{|\mathbf{c}||\mathbf{h} + \delta\mathbf{h}|} \\ \dfrac{\mathbf{h}^\top(\mathbf{r} + \delta\mathbf{r})}{|\mathbf{h}||\mathbf{r} + \delta\mathbf{r}|} & \dfrac{\mathbf{h}^\top(\mathbf{c} + \delta\mathbf{c})}{|\mathbf{h}||\mathbf{c} + \delta\mathbf{c}|} & \dfrac{\mathbf{h}^\top(\mathbf{h} + \delta\mathbf{h})}{|\mathbf{h}||\mathbf{h} + \delta\mathbf{h}|} \end{bmatrix}$$

The vectors $\boldsymbol{\delta}\mathbf{r}$ and $\boldsymbol{\delta}\mathbf{v}$ are the relative position and velocity, respectively, of the chaser with respect to the target in the inertial frame. These can by obtained from $\boldsymbol{\rho}$ and $\boldsymbol{\rho}$', the scaled states in the rotating frame, as

$$\delta\mathbf{r} = C_{\mathrm{target}}^{\top}(r\rho) \tag{44a}$$

$$\begin{aligned}\boldsymbol{\delta}\mathbf{v} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}C_{\mathrm{target}}^{\top}\right)(r\boldsymbol{\rho}) + C_{\mathrm{target}}^{\top}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\boldsymbol{\rho} + r\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}C_{\mathrm{target}}^{\top}\right)(r\boldsymbol{\rho}) + C_{\mathrm{target}}^{\top}(\upsilon_r\boldsymbol{\rho} + \upsilon_\theta\boldsymbol{\rho}')\end{aligned} \tag{44b}$$

The rate of change of the target's direction cosine matrix in equation 44b can be obtained by differentiating equation 41 with respect to time, as [19]

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{\mathrm{target}}^{\top} = \left[\left(\frac{\mathbf{v}}{|\mathbf{r}|} - \frac{(\mathbf{v}^{\top}\mathbf{r})\mathbf{r}}{|\mathbf{r}|^3}\right)\left(\frac{\dot{\mathbf{c}}}{|\mathbf{c}|} - \frac{(\dot{\mathbf{c}}^{\top}\mathbf{c})\mathbf{c}}{|\mathbf{c}|^3}\right)0\right] \tag{45}$$

where $\dot{\mathbf{c}} = \mathbf{h} \times \mathbf{v} = (\mathbf{r} \times \mathbf{v}) \times \mathbf{v}$. It should be noted that because the angular momentum vector of a satellite in a two body orbit is a constant, its time derivative, given by the third column of the matrix in equation 45 is zero. Additionally, it is easy to show for the two-body problem that

$$|\mathbf{h}| = r^2\dot{f} = r\upsilon_\theta \tag{46a}$$

$$|\mathbf{c}| = |\mathbf{h}||\mathbf{r}| = r^2\upsilon_\theta \tag{46b}$$

$$\mathbf{r}^{\top}\mathbf{v} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}^{\top}\mathbf{r} = r\upsilon_r \tag{46c}$$

$$\mathbf{c}^{\top}\mathbf{v} = -\mathbf{r}^{\top}\dot{\mathbf{c}} = \mathbf{v}^{\top}(\mathbf{h}\times\mathbf{r}) = r^2\upsilon_\theta^2 \tag{46d}$$

$$\mathbf{c}^{\top}\dot{\mathbf{c}} = \mathbf{c}^{\top}(\mathbf{h}\times\mathbf{v}) = r^3\upsilon_\theta^2\upsilon_r \tag{46e}$$

$$\mathbf{r}^{\top}\mathbf{c} = \mathbf{r}^{\top}\mathbf{h} = \mathbf{h}^{\top}\mathbf{c} = \mathbf{h}^{\top}\mathbf{v} = \mathbf{v}^{\top}\dot{\mathbf{c}} = \mathbf{h}^{\top}\dot{\mathbf{c}} = 0 \tag{46f}$$

Using equation 44 in equation 43, it can be shown that

$$C_{\mathrm{rel}} = \begin{bmatrix} \dfrac{(|\mathbf{r}|^2 + \mathbf{r}^{\top}\boldsymbol{\delta}\mathbf{r})}{|\mathbf{r}||\mathbf{r}+\boldsymbol{\delta}\mathbf{r}|} & \dfrac{\mathbf{r}^{\top}\boldsymbol{\delta}\mathbf{c}}{|\mathbf{r}||\mathbf{c}+\boldsymbol{\delta}\mathbf{c}|} & \dfrac{\mathbf{r}^{\top}\boldsymbol{\delta}\mathbf{h}}{|\mathbf{r}||\mathbf{h}+\boldsymbol{\delta}\mathbf{h}|}\\ \dfrac{\mathbf{c}^{\top}\boldsymbol{\delta}\mathbf{r}}{|\mathbf{c}||\mathbf{r}+\boldsymbol{\delta}\mathbf{r}|} & \dfrac{(|\mathbf{c}|^2 + \mathbf{c}^{\top}\boldsymbol{\delta}\mathbf{c})}{|\mathbf{c}||\mathbf{c}+\boldsymbol{\delta}\mathbf{c}|} & \dfrac{\mathbf{c}^{\top}\boldsymbol{\delta}\mathbf{h}}{|\mathbf{c}||\mathbf{h}+\boldsymbol{\delta}\mathbf{h}|}\\ \dfrac{\mathbf{h}^{\top}\boldsymbol{\delta}\mathbf{r}}{|\mathbf{h}||\mathbf{r}+\boldsymbol{\delta}\mathbf{r}|} & \dfrac{\mathbf{h}^{\top}\boldsymbol{\delta}\mathbf{c}}{|\mathbf{h}||\mathbf{c}+\boldsymbol{\delta}\mathbf{c}|} & \dfrac{(|\mathbf{h}|^2 + \mathbf{h}^{\top}\boldsymbol{\delta}\mathbf{h})}{|\mathbf{h}||\mathbf{h}+\boldsymbol{\delta}\mathbf{h}|} \end{bmatrix} \tag{47}$$

To obtain the elements of $\boldsymbol{C}_{\mathrm{rel}}$ in terms of the relative states, it is first necessary to define some preliminary quantities. From equations 44 and 45, it is easy to show that

$$\boldsymbol{\delta}\mathbf{r} = rx\hat{\mathbf{r}} + ry\hat{\mathbf{c}} + rz\hat{\mathbf{h}} \tag{48a}$$

$$\boldsymbol{\delta v} = (v_r x - v_\theta y + v_\theta x')\hat{\mathbf{r}} + (v_\theta x + v_r y + v_\theta y')\hat{\mathbf{c}} + (v_r z + v_\theta z')\hat{\mathbf{h}} \tag{48b}$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\hat{\mathbf{r}}|$, $\hat{\mathbf{c}} = \mathbf{c}/|\mathbf{c}|$, and $\hat{\mathbf{h}} = \mathbf{h}/|\mathbf{h}|$.

The numerators of the elements in the first column of equation 47 can be resolved by using equations 48a and 46, as

$$\mathbf{r}^\top \boldsymbol{\delta}\mathbf{r} = r^2 x \tag{49a}$$

$$\mathbf{c}^\top \boldsymbol{\delta}\mathbf{r} = r^3 v_\theta y \tag{49b}$$

$$\mathbf{h}^\top \boldsymbol{\delta}\mathbf{r} = r^2 v_\theta z \tag{49c}$$

The numerators of the elements in the third column of equation 47 are then resolved by noting that

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \mathbf{v} \\ \Rightarrow \boldsymbol{\delta}\mathbf{h} &= \boldsymbol{\delta}\mathbf{r} \times \mathbf{v} + \mathbf{r} \times \boldsymbol{\delta}\mathbf{v} + \boldsymbol{\delta}\mathbf{r} \times \boldsymbol{\delta}\mathbf{v} \end{aligned} \tag{50}$$

where,

$$\boldsymbol{\delta}\mathbf{r} \times \mathbf{v} = -r v_\theta z\, \hat{\mathbf{r}} + r v_r z\, \hat{\mathbf{c}} + r(v_\theta x - v_r y)\hat{\mathbf{h}} \tag{51a}$$

$$\mathbf{r} \times \boldsymbol{\delta}\mathbf{v} = -r(v_r z + v_\theta z')\hat{\mathbf{c}} + r(v_\theta x + v_r y + v_\theta y')\hat{\mathbf{h}} \tag{51b}$$

$$\begin{aligned} \boldsymbol{\delta}\mathbf{r} \times \boldsymbol{\delta}\mathbf{v} &= r v_\theta[-xz\hat{\mathbf{r}} - yz\hat{\mathbf{c}} + (x^2 + y^2)\hat{\mathbf{h}}] \\ &\quad + r v_\theta[(yz' - zy')\hat{\mathbf{r}} + (zx' - xz')\hat{\mathbf{c}} + (xy' - yx')\hat{\mathbf{h}}] \end{aligned} \tag{51c}$$

Therefore, the quantity $\boldsymbol{\delta}\mathbf{h}$ is given in terms of relative states by

$$\begin{aligned} \boldsymbol{\delta}\mathbf{h} &= r v_\theta[(-z - xz + yz' - zy')\hat{\mathbf{r}} + (-z' - yz + zx' - xz')\hat{\mathbf{c}} \\ &\quad + (2x + y' + x^2 + y^2 + xy' - yx')\hat{\mathbf{h}}] \end{aligned} \tag{52}$$

and

$$\mathbf{r}^\top \boldsymbol{\delta}\mathbf{h} = r^2 v_\theta(-z - xz + yz' - zy') \tag{53a}$$

$$\mathbf{c}^\top \boldsymbol{\delta}\mathbf{h} = r^3 v_\theta^2(-z' - yz + zx' - xz') \tag{53b}$$

$$\mathbf{h}^\top \boldsymbol{\delta}\mathbf{h} = r^2 v_\theta^2(2x + y' + x^2 + y^2 + xy' - yx') \tag{53c}$$

Calculating the elements of the second column of the relative direction cosine matrix requires the vector $\boldsymbol{\delta c}$ in terms of the relative states. This vector is given by

$$\begin{aligned} \boldsymbol{\delta}\mathbf{c} &= -r^2 v_\theta[y + x(2y - x') + (xx' + yy' + zz') + (y - x')(x^2 + y^2 + z^2) \\ &\quad + x(xx' + yy' + zz')]\hat{\mathbf{r}} + r^2 v_\theta[3x + y' + x(2x + y') + (xy' - yx') \\ &\quad + (x^2 + y^2 + z^2) + (x + y')(x^2 + y^2 + z^2) - y(xx' + yy' + zz')]\hat{\mathbf{c}} \end{aligned}$$

$$+ r^2 v_\theta[z' + 2xz' - zx' + z'(x^2 + y^2 + z^2) - z(xx' + yy' + zz')]\hat{\mathbf{h}} \quad (54)$$

$$\begin{aligned}\mathbf{r}^\top \boldsymbol{\delta}\mathbf{c} &= -r^3 v_\theta[y + x(2y - x') + (xx' + yy' + zz') \\ &\quad + (y - x')(x^2 + y^2 + z^2) + x(xx' + yy' + zz')] \quad (55a)\end{aligned}$$

$$\begin{aligned}\mathbf{c}^\top \boldsymbol{\delta}\mathbf{c} &= r^4 v_\theta^2[3x + y' + x(2x + y') + (xy' - yx') + (x^2 + y^2 + z^2) \\ &\quad + (x + y')(x^2 + y^2 + z^2) - y(xx' + yy' + zz')] \quad (55b)\end{aligned}$$

$$\mathbf{h}^\top \boldsymbol{\delta}\mathbf{c} = r^3 v_\theta^2[z' + 2xz' - zx' + z'(x^2 + y^2 + z^2) - y(xx' + yy' + zz')] \quad (55c)$$

The denominators of the terms in equation 43 can be written in terms of the states. Because the vectors **r**, **c**, **h** form an orthogonal basis for the state space

$$\begin{aligned}\boldsymbol{\delta}\mathbf{r} &= \frac{\mathbf{r}^\top \boldsymbol{\delta}\mathbf{r}}{|\mathbf{r}|}\hat{\mathbf{r}} + \frac{\mathbf{c}^\top \boldsymbol{\delta}\mathbf{r}}{|\mathbf{c}|}\hat{\mathbf{c}} + \frac{\mathbf{h}^\top \boldsymbol{\delta}\mathbf{r}}{|\mathbf{h}|}\hat{\mathbf{h}} \\ \Rightarrow \boldsymbol{\delta}\mathbf{r}^\top \boldsymbol{\delta}\mathbf{r} &= \frac{(\mathbf{r}^\top \boldsymbol{\delta}\mathbf{r})^2}{r^2} + \frac{(\mathbf{c}^\top \boldsymbol{\delta}\mathbf{r})^2}{r^4 v_\theta^2} + \frac{(\mathbf{h}^\top \boldsymbol{\delta}\mathbf{r})^2}{r^2 v_\theta^2} \\ &= r^2|\boldsymbol{\rho}|^2 \quad (56a)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\delta}\mathbf{c} &= \frac{\mathbf{r}^\top \boldsymbol{\delta}\mathbf{c}}{|\mathbf{r}|}\hat{\mathbf{r}} + \frac{\mathbf{c}^\top \boldsymbol{\delta}\mathbf{c}}{|\mathbf{c}|}\hat{\mathbf{c}} + \frac{\mathbf{h}^\top \boldsymbol{\delta}\mathbf{c}}{|\mathbf{h}|}\hat{\mathbf{h}} \\ \Rightarrow \boldsymbol{\delta}\mathbf{c}^\top \boldsymbol{\delta}\mathbf{c} &= \frac{(\mathbf{r}^\top \boldsymbol{\delta}\mathbf{c})^2}{r^2} + \frac{(\mathbf{c}^\top \boldsymbol{\delta}\mathbf{c})^2}{r^4 v_\theta^2} + \frac{(\mathbf{h}^\top \boldsymbol{\delta}\mathbf{c})^2}{r^2 v_\theta^2} \quad (56b)\end{aligned}$$

$$\begin{aligned}\boldsymbol{\delta}\mathbf{h} &= \frac{\mathbf{r}^\top \boldsymbol{\delta}\mathbf{h}}{|\mathbf{r}|}\hat{\mathbf{r}} + \frac{\mathbf{c}^\top \boldsymbol{\delta}\mathbf{h}}{|\mathbf{c}|}\hat{\mathbf{c}} + \frac{\mathbf{h}^\top \boldsymbol{\delta}\mathbf{h}}{|\mathbf{h}|}\hat{\mathbf{h}} \\ \Rightarrow \boldsymbol{\delta}\mathbf{h}^\top \boldsymbol{\delta}\mathbf{h} &= \frac{(\mathbf{r}^\top \boldsymbol{\delta}\mathbf{h})^2}{r^2} + \frac{(\mathbf{c}^\top \boldsymbol{\delta}\mathbf{h})^2}{r^4 v_\theta^2} + \frac{(\mathbf{h}^\top \boldsymbol{\delta}\mathbf{h})^2}{r^2 v_\theta^2} \quad (56c)\end{aligned}$$

Using equations 49, 53, 55, and 56, the entries of C_{rel} are

$$C_{\mathrm{rel}}(1,1) = (1 + x)/[(1 + x)^2 + y^2 + z^2]^{-1/2} \quad (57a)$$

$$C_{\mathrm{rel}}(2,1) = y/[(1 + x)^2 + y^2 + z^2]^{-1/2} \quad (57b)$$

$$C_{\mathrm{rel}}(3,1) = z/[(1 + x)^2 + y^2 + z^2]^{-1/2} \quad (57c)$$

$$\begin{aligned}C_{\mathrm{rel}}(1,2) &= -[y + x(2y - x') + (xx' + yy' + zz') + (y - x')(x^2 + y^2 + z^2) \\ &\quad + x(xx' + yy' + zz')]/\{[y + x(2y - x') + (xx' + yy' + zz')\end{aligned}$$

$$+ (y - x') + (x^2 + y^2 + z^2) + x(xx' + yy' + zz')]^2 + [1 + 3x + y'$$

$$+ x(2x + y') + (xy' - yx') + (x^2 + y^2 + z^2) + (x + y')(x^2 + y^2 + z^2)$$

$$- y(xx' + yy' + zz')]^2 + [z' + 2xz' - zx' + z'(x^2 + y^2 + z^2)$$

$$- z(xx' + yy' + zz')]^2\}^{-1/2} \tag{57d}$$

$$C_{\mathrm{rel}}(2,2) = [1 + 3x + y' + x(2x + y') + (xy' - yx') + (x^2 + y^2 + z^2)$$

$$+ (x + y')(x^2 + y^2 + z^2) - y(xx' + yy' + zz')]/\{[y + x(2y - x')$$

$$+ (xx' + yy' + zz') + (y - x')(x^2 + y^2 + z^2) + x(xx' + yy' + zz')]^2$$

$$+ [1 + 3x + y' + x(2x + y') + (xy' - yx') + (x^2 + y^2 + z^2)$$

$$+ (x + y')(x^2 + y^2 + z^2) - y(xx' + yy' + zz')]^2 + [z' + 2xz' - zx'$$

$$+ z'(x^2 + y^2 + z^2) - z(xx' + yy' + zz')]^2\}^{-1/2} \tag{57e}$$

$$C_{\mathrm{rel}}(3,2) = [z' + 2xz' - zx' + z'(x^2 + y^2 + z^2) - z(xx' + yy' + zz')]/$$

$$\{[y + x(2y - x') + (xx' + yy' + zz') + (y - x')(x^2 + y^2 + z^2)$$

$$+ x(xx' + yy' + zz')]^2 + [1 + 3x + y' + x(2x + y') + (xy' - yx')$$

$$+ (x^2 + y^2 + z^2) + (x + y')(x^2 + y^2 + z^2) - y(xx' + yy' + zz')]^2$$

$$+ [z' + 2xz' - zx' + z'(x^2 + y^2 + z^2) - z(xx' + yy' + zz')]^2\}^{-1/2} \tag{57f}$$

$$C_{\mathrm{rel}}(1,3) = -(z + xz - yz' + zy')/[(z + xz - yz' + zy')^2 + (z' + yz - zx' + xz')^2$$

$$+ (1 + 2x + y' + x^2 + y^2 + xy' - yx')^2]^{-1/2} \tag{57g}$$

$$C_{\mathrm{rel}}(2,3) = -(z' + yz - zx' + xz')/[(z + xz - yz' + zy')^2 + (z' + yz - zx' + xz')^2$$

$$+ (1 + 2x + y' + x^2 + y^2 + xy' - yx')^2]^{-1/2} \tag{57h}$$

$$C_{\mathrm{rel}}(3,3) = (1 + 2x + y' + x^2 + y^2 + xy' - yx')/[(z + xz - yz' + zy')^2$$

$$+ (z' + yz - zx' + xz')^2$$

$$+ (1 + 2x + y' + x^2 + y^2 + xy' - yx')^2]^{-1/2} \tag{57i}$$

Equation 57 compose the complete, nonlinear relative direction cosine matrix. By noting that the magnitudes of relative states are always less than unity, and for most purposes, are several orders of magnitude less than unity, equation 57 can be written as a series of polynomials of increasing order. The matrix C_{rel} can then be written in polynomial form, truncated at the appropriate order. The denominators in equation 56 may be rewritten

$$\begin{aligned} |\mathbf{r} + \boldsymbol{\delta}\mathbf{r}|^{-1} &= [(\mathbf{r} + \boldsymbol{\delta}\mathbf{r})^\top(\mathbf{r} + \boldsymbol{\delta}\mathbf{r})]^{-1/2} \\ &= (\mathbf{r}^\top\mathbf{r} + 2\mathbf{r}^\top\,\boldsymbol{\delta}\mathbf{r} + \boldsymbol{\delta}\mathbf{r}^\top\boldsymbol{\delta}\mathbf{r})^{-1/2} \\ &= \frac{1}{|\mathbf{r}|}\left(1 + 2\frac{\mathbf{r}^\top\boldsymbol{\delta}\mathbf{r}}{|\mathbf{r}||\boldsymbol{\delta}\mathbf{r}|}\cdot\frac{|\boldsymbol{\delta}\mathbf{r}|}{|\mathbf{r}|} + \frac{|\boldsymbol{\delta}\mathbf{r}|^2}{|\mathbf{r}|^2}\right)^{-1/2} \\ &= \frac{1}{r}\sum_{k=0}^{\infty}(-1)^k\left(\frac{|\boldsymbol{\delta}\mathbf{r}|}{|\mathbf{r}|}\right)^k P_k\left(\frac{\mathbf{r}^\top\boldsymbol{\delta}\mathbf{r}}{|\mathbf{r}||\boldsymbol{\delta}\mathbf{r}|}\right) \end{aligned} \tag{58a}$$

$$\text{Similarly,}\quad |\mathbf{c} + \boldsymbol{\delta}\mathbf{c}|^{-1} = \frac{1}{r^2 v_\theta}\sum_{k=0}^{\infty}(-1)^k\left(\frac{|\boldsymbol{\delta}\mathbf{c}|}{|\mathbf{c}|}\right)^k P_k\left(\frac{\mathbf{c}^\top\boldsymbol{\delta}\mathbf{c}}{|\mathbf{c}||\boldsymbol{\delta}\mathbf{c}|}\right) \tag{58b}$$

$$\text{and,}\ |\mathbf{h} + \boldsymbol{\delta}\mathbf{h}|^{-1} = \frac{1}{r v_\theta}\sum_{k=0}^{\infty}(-1)^k\left(\frac{|\boldsymbol{\delta}\mathbf{h}|}{|\mathbf{h}|}\right)^k P_k\left(\frac{\mathbf{h}^\top\boldsymbol{\delta}\mathbf{h}}{|\mathbf{h}||\boldsymbol{\delta}\mathbf{h}|}\right) \tag{58c}$$

where P_k is the kth Legendre polynomial. Equation 58 can be further simplified into polynomials of the state variables. The summand in each of equation 58 can be written as

$$\left(\frac{|\boldsymbol{\delta}\mathbf{r}|}{|\mathbf{r}|}\right)^k P_k\left(\frac{\mathbf{r}^\top\boldsymbol{\delta}\mathbf{r}}{|\mathbf{r}||\boldsymbol{\delta}\mathbf{r}|}\right) = (x^2 + y^2 + z^2)^{k/2}\, P_k\left(\frac{x}{(x^2 + y^2 + z^2)^{1/2}}\right) \tag{59a}$$

$$\left(\frac{|\delta\mathbf{c}|}{|\mathbf{c}|}\right)^k P_k\left(\frac{\mathbf{c}^\top\delta\mathbf{c}}{|\mathbf{c}||\delta\mathbf{c}|}\right) = (\delta c_x^2 + \delta c_y^2 + \delta c_z^2)^{k/2}\, P_k\left(\frac{\delta c_y}{(\delta c_x^2 + \delta c_y^2 + \delta c_z^2)^{1/2}}\right) \tag{59b}$$

$$\left(\frac{|\delta\mathbf{h}|}{\mathbf{h}}\right)^k P_k\left(\frac{\mathbf{h}^\top\delta\mathbf{h}}{|\mathbf{h}||\delta\mathbf{h}|}\right) = (\delta h_x^2 + \delta h_y^2 + \delta h_z^2)^{k/2}\, P_k\left(\frac{\delta h_z}{(\delta h_x^2 + \delta h_y^2 + \delta h_z^2)^{1/2}}\right) \tag{59c}$$

where $\delta c_x\hat{\mathbf{r}} + \delta c_y\hat{\mathbf{c}} + \delta c_z\hat{\mathbf{h}} = \boldsymbol{\delta}\mathbf{c}/r^2 v_\theta$ and $\delta h_x\hat{\mathbf{r}} + \delta h_y\hat{\mathbf{c}} + \delta h_z\hat{\mathbf{h}} = \boldsymbol{\delta}\mathbf{h}/r v_\theta$.

Using equation 58, the relative direction cosine matrix can be written as

$$C_{\mathrm{rel}} = \begin{bmatrix} 1 - (y^2 + z^2)/2 & -y + xy - zz' & -z + yz' + xz \\ y - xy & 1 - (y^2 + z'^2)/2 & -z' + (x + y')z' + (x' - y)z \\ z - xz & z' - (x + y')z' - zx' & 1 - (z^2 + z'^2)/2 \end{bmatrix}$$

$$+ \text{h.o.t} \tag{60}$$

Nomenclature

a = Semimajor axis

e = Eccentricity

f = True anomaly

i = Inclination

p = Semiparameter

$P_k(s)$ = Legendre polynomial of order k and argument s

r = Radial distance of target satellite from the planet

$(u_{\xi,}\ u_{\vartheta},\ u_{\zeta})$ = Components of control in the radial, along-track, and out-of-plane position

$(u_r,\ u_\theta,\ u_h)$ = Scaled components of control in the radial, along-track, and out-of-plane position

v_r = Radial velocity of the target satellite

v_θ = Circumferential velocity of the target satellite

μ = Gravitational constant of the planet

$\rho = \{x \quad y \quad z\}^\top$ = Scaled relative position vector with components along radial, along-track, out-of-plane direction

$\varrho = \{\xi \quad \vartheta \quad \zeta\}^\top$ = Relative position vector with components along radial, along-track, out-of-plane direction

θ = Argument of latitude

Ω = Right Ascension of Ascending Node (RAAN)

ω = Argument of perigee

$(')$ = Derivative with respect to true anomaly

$(\dot{})$ = Derivative with respect to time

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