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The Lambert W Function and Solutions to Kepler's Equation

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Abstract This paper presents a method for the truncation of infinite Fourier-Bessel representations for functions requiring a solution to Kepler's equation. Use is made of the Lambert W function to solve for the desired index that bounds the remainder terms of the series, within the prescribed tolerance. The enforcement of a maximum on the number of Bessel functions is also useful in truncating the Bessel functions themselves, resulting in an analytical representation of the solution to a desired tolerance, without the use of infinite series.

Keywords Kepler's equation · Lambert W function · series solution

1 Introduction

The Lambert W function (Corless et al. 1996) is defined as the multivalued inverse of the following function:

$$\mathcal{W}(z) \exp[\mathcal{W}(z)] = z \tag{1}$$

As shown by Corless et al. (1996), this function has several uses in physical and engineering applications. Recent work by Galidakis (2004) shows how the W function also has uses in testing convergence properties of infinite exponentials. It is therefore a natural extension that several results for the W function can also be used in series solutions resulting from an analysis of Kepler's equation (Colwell 1993).

2 Keplerian Anomalies and Kepler's Equation

The Kepler anomalies are quantities that are used to calculate the position of an orbiting body, in a central gravity field. The equation of the conic is given by $r = a\eta^2/(1 + e\cos f)$, where r is the current radial distance of the satellite, a is the semimajor axis, $\eta = \sqrt{1-e^2}$, e is the eccentricity of the orbit, and f is the *true* anomaly. The conic equation is also given by $r = a(1 - e\cos E)$, where E is the *eccentric* anomaly. The true and eccentric anomalies are related by the following equation (Battin 1999):

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{2}$$

The satellite's orbit is dependent on time since epoch through the *mean* anomaly, $M = M_0 + n\,\Delta t$, where M_0 is the mean anomaly at epoch, $n = \sqrt{\mu/a^3}$, and Δt is the elapsed time. However, the

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dependence is not explicit, and is given by the following transcendental equation, known as Kepler's equation:

$$M = E - e \sin E \tag{3}$$

Equation (3) has no known closed-form solution. However, this equation is amenable to numerical methods and solutions based on infinite series, the details of which are given in Colwell (1993). In some cases, a function is used to represent the iterative solution to Kepler's equation; for example, the nesting function used by Ketema (2005). However, the numerical methods do not lend themselves readily to characterization of the solution in a functional form, while the series solutions, at the first glance, appear to require an infinite number of terms, which are computationally impossible to generate. Mathematical operations also cannot be performed on the nesting function, since it represents an iterative procedure to obtain the solution to Kepler's equation and not the solution itself.

Many engineering applications related to orbital mechanics require time-explicit formulation of quantities that are also valid for arbitrary eccentricities. For example, the STMs for relative motion derived by Carter (1998); Yamanaka and Ankersen (2002); Sengupta et al. (2007) are formulated using the true anomaly, and for application purposes, require a conversion from time (mean anomaly) to true anomaly, using numerical techniques. A series representation of the functions of true anomaly, in terms of mean anomaly, would result in and infinite number of terms with coefficients based on Bessel functions. While the series may be terminated, it is not known *a priori* how many terms are required, or at what point the series may be terminated, without checking for convergence within numerical tolerance.

Typically, series solutions are truncated at a prescribed order of eccentricity. For example, Melton (2000) derived a time-explicit state transition matrix (STM) for relative motion, that was correct through the second order in eccentricity. These methods are also used in cases other than two-body motion; for example, Richardson and Cary (1975) employed a first-order expansion in eccentricity to design Lissajous orbits around the collinear libration points in the elliptic restricted three-body problem. However, truncating the series at a given order of eccentricity does not reveal the order of the resulting error, or the region of validity, unless compared with numerical simulations that include the full effects of eccentricity.

In this paper, it is shown that the Lambert W function can be used to truncate series solutions to Kepler's equation. In particular, the minimum number of terms required to reduce the contribution of higher order terms within the desired tolerance, is obtained from the solution of this function. Furthermore, the accuracy of the solutions when terms beyond a certain order of eccentricity are ignored, can also be obtained. It is worth noting that the number of operations required to calculate series solutions can be very large in comparison to most modern-day algorithms (Mortari and Clocchiatti 2006). While for the combination of moderate-to-high eccentricity and high numerical accuracy the series solutions (infinite or finite) are not expected to be particularly useful, they can provide qualitative expressions for algebraic manipulation.

3 Series Representation and Truncation

In this section, the theory used to obtain the upper bound is outlined, by demonstrating the procedure on an example. The technique is easily extended to obtain the expansion of any function of the true or eccentric anomalies in terms of the mean anomaly.

3.1 Expansion of the Eccentric Anomaly

The series expansion of the eccentric anomaly is considered. It is known that (Colwell 1993; Battin 1999):

$$E = M + 2 \sum_{k=1}^{\infty} \frac{1}{k} J_k(ke) \sin kM \tag{4}$$

where J_k is the kth-order Bessel function of the first kind (Abramowitz and Stegun 1972), given by the following series:

$$J_k(x) = \sum_{j=0}^{\infty}(-1)^j \frac{(x/2)^{k+2j}}{j!\,(k+j)!} \tag{5}$$

Watson (1966) has shown that:

$$|J_\nu(\nu x)| \leq \frac{\exp\left[-\nu\, F(0,x)\right]}{(1-x^2)^{1/4}(2\pi\nu)^{1/2}} \tag{6}$$

where

$$F(\theta,x) = \ln\left[\frac{\theta + (\theta^2 - x^2\,\sin^2\theta)^{1/2}}{x\sin\theta}\right] - (\theta^2 - x^2\,\sin^2\theta)^{1/2}\,\cot\theta \tag{7}$$

Since $F(\theta,x)$ is continuous, (7) may be evaluated at $\theta = 0$ by calculating the limit as shown below:

$$F(0,x) = \lim_{\theta\to 0} F(\theta,x) = \ln\left[\frac{1+\sqrt{1-x^2}}{x}\right] - \sqrt{1-x^2} \tag{8}$$

Using $\nu = k$, $x = e$, and (8) in (6) and (7), the following inequality is obtained:

$$|J_k(ke)| \leq \frac{\varepsilon^k\,\exp(k\eta)}{\sqrt{2\pi\eta\,k}} \tag{9}$$

where

$$\eta = \sqrt{1-e^2} \tag{10a}$$

$$\varepsilon = \sqrt{\frac{1-\eta}{1+\eta}} \tag{10b}$$

Let $\xi = \varepsilon\,\exp\eta$. It can be shown that $0 \leq \xi \leq 1$, by using the following result:

$$\ln\left(\varepsilon\,\exp\eta\right) = \frac{1}{2}\left[\ln(1-\eta) - \ln(1+\eta)\right] + \eta = -\left[\frac{1}{3}\eta^3 + \frac{1}{5}\eta^5 + \cdots\right]$$

$$\text{or,}\quad -\infty \leq \ln\xi \leq 0 \tag{11}$$

Equation (9) is the basic inequality used to find truncations to Fourier-Bessel series. Let $k_{\max}$ be the index at which the series in (4) is truncated. Consequently, the sum of the terms with indices $k > k_{\max}$ must satisfy the following inequality:

$$\left|2\sum_{k=k^*}^{\infty}\frac{1}{k}J_k(ke)\,\sin kM\right| \leq 10^{-N_{\text{tol}}} \tag{12}$$

where $k^* = k_{\max}+1$, and $N_{\text{tol}} \in \mathbb{Z}^+$ is a number indicating the desired numerical tolerance. Using (9), the sum of the series in (12) is bounded as follows:

$$\begin{aligned}\left|2\sum_{k=k^*}^{\infty}\frac{1}{k}J_k(ke)\,\sin kM\right| &\leq 2\sum_{k=k^*}^{\infty}\frac{1}{k}\,|J_k(ke)| \\ &\leq \frac{2}{\sqrt{2\pi\eta}}\sum_{k=k^*}^{\infty}\frac{\xi^k}{k^{3/2}} \\ &= \frac{2}{\sqrt{2\pi\eta}}\xi^{k^*}\left[\frac{1}{k^{*3/2}} + \frac{\xi}{(k^*+1)^{3/2}} + \frac{\xi^2}{(k^*+2)^{3/2}} + \cdots\right] \\ &\leq \frac{2}{\sqrt{2\pi\eta}}\xi^{k^*}\left[\frac{1}{k^{*3/2}} + \frac{\xi}{k^{*3/2}} + \frac{\xi^2}{k^{*3/2}} + \cdots\right]\end{aligned}$$

$$\text{or,} \quad \left| 2 \sum_{k=k^*}^{\infty} \frac{1}{k} J_k(ke) \sin kM \right| \leq \frac{2}{\sqrt{2\pi\eta}} \frac{\xi^{k^*}}{{k^*}^{3/2}} \frac{1}{(1-\xi)} \tag{13}$$

The value for the index k^* may then be obtained by solving the following equation:

$$\frac{2}{\sqrt{2\pi\eta}} \frac{\xi^{k^*}}{{k^*}^{3/2}} \frac{1}{(1-\xi)} = 10^{-N_{\text{tol}}} \tag{14}$$

Applying the natural logarithmic operator to both sides, the following equation is obtained:

$$c_1 k^* + c_2 \ln k^* = c_3 \tag{15}$$

where

$$c_1 = -\ln \xi \tag{16a}$$

$$c_2 = \frac{3}{2} \tag{16b}$$

$$c_3 = N_{\text{tol}} \ln 10 + \ln \frac{2}{\sqrt{2\pi\eta}(1-\xi)} \tag{16c}$$

Equation (15) is rewritten as:

$$\left(\frac{c_1}{c_2} k^*\right) \exp\left(\frac{c_1}{c_2} k^*\right) = \frac{c_1}{c_2} \exp\left(\frac{c_3}{c_2}\right) \tag{17}$$

In (17), let $(c_1/c_2)\exp(c_3/c_2) = z$, and $(c_1/c_2)\, k^* = \mathcal{W}(z)$. Comparing (17) with (1), it follows that k^* can be obtained from the Lambert W function. Since only integer values of k^* are of interest, the following is obtained as the solution to $k_{\max}$:

$$k_{\max} = \lceil k^* \rceil - 1 = \left\lceil \left(\frac{c_2}{c_1}\right) \mathcal{W}\left(\exp\left(\frac{c_3}{c_2}\right) \frac{c_1}{c_2}\right) \right\rceil - 1 \tag{18}$$

where $\lceil \cdot \rceil$ denotes the ceiling function. Consequently, if an analytical representation for E, in terms of M, is required, such that $|E - M| \leq 10^{-N_{\text{tol}}}$, then,

$$E \approx M + 2 \sum_{k=1}^{k_{\max}} \frac{1}{k} J_k(ke) \sin kM \tag{19}$$

where $k_{\max}$ is given by (17) and (16), and is dependent only on the desired tolerance N_{tol} and the eccentricity e. Equation (19) allows operations such as symbolic integration or differentiation, and is also numerically exact within the desired tolerance.

3.2 Generalization of the Result

Series expansions of E or f in terms of M are typically composed of either $J_k(ke)$ or $J'_k(ke)$, where $(\,'\,)$ denotes a derivative with respect to e, and harmonics of kM. Therefore, the following two series are considered:

$$a = q \sum_{k=1}^{\infty} \frac{1}{k^p} J_k(ke) \exp(\imath kM) \tag{20a}$$

$$b = q \sum_{k=1}^{\infty} \frac{1}{k^p} J'_k(ke) \exp(\imath kM) \tag{20b}$$

where $\imath = \sqrt{-1}$, $p, q \in \mathbb{R}$, $p \geq 0$, and $q > 0$. For example, the expansions of $\cos f$ and $\sin f$ in terms of the mean anomaly, as shown in Battin (1999), are special cases of (20a) and (20b), respectively:

$$\cos f = -e + \frac{2\eta^2}{e} \sum_{k=1}^{\infty} J_k(ke) \cos kM \tag{21a}$$

$$\sin f = 2\eta \sum_{k=1}^{\infty} \frac{1}{k} \frac{d}{de} J_k(ke) \sin kM \tag{21b}$$

Let a_k and b_k denote the kth term of the series in (20). From Watson (1966), a bound for $J_k'(ke)$ is obtained as follows:

$$|J_k'(ke)| \le \frac{(1+e^2)^{1/4}}{e\sqrt{2\pi}} \sqrt{k}\, \xi^k \tag{22}$$

From (9) and (22), it follows that

$$a_k^* \triangleq \left| \sum_{k=k^*}^{\infty} a_k \right| \le \frac{|q|}{\sqrt{2\pi\eta}} \sum_{k=k^*}^{\infty} \frac{\xi^k}{k^{(p+1/2)}} \le \frac{|q|}{\sqrt{2\pi\eta}(1-\xi)} \frac{\xi^{k^*}}{k^{*(p+1/2)}} \tag{23a}$$

$$b_k^* \triangleq \left| \sum_{k=k^*}^{\infty} b_k \right| \le \frac{|q|(1+e^2)^{1/4}}{e\sqrt{2\pi\eta}} \sum_{k=k^*}^{\infty} \frac{\xi^k}{k^{(p-1/2)}} \le \frac{|q|(1+e^2)^{1/4}}{e\sqrt{2\pi\eta}(1-\xi)} \frac{\xi^{k^*}}{k^{*(p-1/2)}} \tag{23b}$$

The maximum index for truncation is obtained by solving either $a_k^* = 10^{-N_{\text{tol}}}$ or $b_k^* = 10^{-N_{\text{tol}}}$, depending on the series used. This results in the following equation:

$$c_e k^* + c_p \ln k^* = c_N \tag{24}$$

where

$$c_e = -\ln \xi \tag{25a}$$

$$c_p = \begin{cases} \left(p + \frac{1}{2}\right), & a_k^* = 10^{-N_{\text{tol}}} \\ \left(p - \frac{1}{2}\right), & b_k^* = 10^{-N_{\text{tol}}} \end{cases} \tag{25b}$$

$$c_N = \begin{cases} N_{\text{tol}} \ln 10 - \ln(1-\xi) + \ln\left(|q|/\sqrt{2\pi\eta}\right), & a_k^* = 10^{-N_{\text{tol}}} \\ N_{\text{tol}} \ln 10 - \ln(1-\xi) + \ln\left(|q|(1+e^2)^{1/4}/\sqrt{2\pi e^2}\right), & b_k^* = 10^{-N_{\text{tol}}} \end{cases} \tag{25c}$$

It should be noted that when $p = 1/2$ for the series a, or $p = -1/2$ for the series b, k_{max} in both cases is given trivially by:

$$k_{\text{max}} = \lceil k^* \rceil - 1 = \left\lceil \left(\frac{c_N}{c_e} \right) \right\rceil - 1 \tag{26}$$

In all other cases

$$k_{\text{max}} = \left\lceil \left(\frac{c_p}{c_e} \right) \mathcal{W} \left(\exp\left(\frac{c_N}{c_p} \right) \frac{c_e}{c_p} \right) \right\rceil - 1 \tag{27}$$

Corless et al. (1997) present several methods to evaluate the Lambert W function in an efficient manner; however, a second-order Newton-Raphson correction, that is found sufficient for $q = 1$, $e \le 0.99$, $p < 4$, and $N_{\text{tol}} \le 15$, is given by:

$$k_{\text{max}} = \left\lceil \frac{c_N}{c_e} \left\{ 1 - \frac{2c_p(c_N + c_p) \ln(c_N/c_e)}{\left[2(c_N + c_p)^2 + c_p^2 \ln(c_N/c_e)\right]} \right\} \right\rceil - 1 \tag{28}$$

3.3 Truncation of Bessel Functions

The truncation introduced by the index $k_{\max}$ to reduce errors within a prescribed tolerance also suggests that the Bessel functions $J_k(ke)$ need not be evaluated as an infinite series. Following the approach in the previous section, the magnitude of terms comprising $j > s$ in (5) is considered. The truncated Bessel function $J_k^s(x)$ is defined as follows:

$$J_k^s(x) = \sum_{j=0}^{s} (-1)^j \frac{(x/2)^{k+2j}}{j!\,(k+j)!} \tag{29}$$

Since the Bessel function (and its derivatives) is a hypergeometric series with alternating sign, the remainder due to truncation at $j = s$ is bounded by the magnitude of the $(s+1)$th term (Du et al. 2002):

$$\left| \sum_{j=s^*}^{\infty} (-1)^j \frac{(x/2)^{k+2j}}{j!\,(k+j)!} \right| \leq \frac{(x/2)^{k+2s^*}}{s^*!\,(k+s^*)!} \tag{30}$$

where $s^* = s + 1$. From Stirling's approximation,

$$s^*! \approx \sqrt{2\pi} s^{*(s^*+1/2)} \exp(-s^*) \tag{31a}$$

$$(k+s^*)! \approx \sqrt{2\pi}(k+s^*)^{(k+s^*+1/2)} \exp(-k-s^*) \tag{31b}$$

Using the above equation, the inequality on the remainder due to truncation of $J_k(ke)$ is given by:

$$\left| \sum_{j=s^*}^{\infty} (-1)^j \frac{(ke/2)^{k+2j}}{j!\,(k+j)!} \right| \leq \frac{1}{2\pi} \frac{(ke/2)^{k+2s^*}}{s^{*(s^*+1/2)}\,(k+s^*)^{(k+s^*+1/2)}} \exp(k+2s^*) \tag{32}$$

A value for s^* maybe obtained by applying the logarithmic operator to the right hand side of the above equation to obtain the following:

$$g(s^*) = b_1 + b_2 s^* - \left(s^* + \frac{1}{2}\right) \ln s^* - \left(k + s^* + \frac{1}{2}\right) \ln(k+s^*) = 0 \tag{33}$$

where

$$b_1 = -\ln(2\pi) + k \ln\left(\frac{ke}{2}\right) + k + N_{\text{tol}} \ln 10 \tag{34a}$$

$$b_2 = 2 \ln\left(\frac{ke}{2}\right) + 2 \tag{34b}$$

It is clear that the number of terms required, s depends on the order of the Bessel function, k. The convergence rate of the Bessel function decreases with its order, and since the number of Bessel functions required is already restricted to $1 \leq k \leq k_{\max}$, $k = k_{\max}$ is substituted in (33) to obtain the highest order of eccentricity required.

Although (33) does not have a closed form solution, $s^* \in \mathbb{Z}$, and consequently a second-order Newton-Raphson solution with an initial guess of $s^* = k_{\max}/2$ is found sufficient for accurate values. Therefore, the order of eccentricity to which Bessel function expansions are required, can be shown to be the following:

$$s = \left\lceil (k_{\max}/2) + 2\frac{g_1(k_{\max}/2)}{g_2(k_{\max}/2)} \right\rceil - 1 \tag{35}$$

$$\text{where,} \quad g_1(s^*) = \frac{d}{ds^*}\left(\frac{1}{g(s^*)}\right)$$

$$g_2(s^*) = \frac{d^2}{ds^{*2}}\left(\frac{1}{g(s^*)}\right)$$

(36)

Similarly, the derivative of the kth-order Bessel function, truncated at $j = s$, is bounded by the $(s+1)$th term, and a solution for s^* can then be obtained from the following equation:

$$\frac{1}{2\pi}\frac{k}{2e}(k+2s^*)\,\exp(k+2s^*)\,\frac{(ke/2)^{(k+2s^*)}}{s^{*(s^*+1/2)}\,(k+s^*)^{(k+s^*+1/2)}} = 10^{-N_{\text{tol}}} \tag{37}$$

4 Examples

As the first example, the series expansion of $\cos f$ given by (21a) is considered. A truncated version of this series is desired, that is valid for $e \leq 0.1$ and a tolerance of $N_{\text{tol}} = 9$. Substituting $p = 1$, $q = 2\eta^2/e = 19.8$, and $N_{\text{tol}} = 9$ in (25) and (28) results in $k_{\max} = 9$. Therefore,

$$\cos f \approx -e + \frac{2\eta^2}{e}\sum_{k=1}^{9} J_k(ke)\,\cos kM \tag{38}$$

As a second approximation, the evaluation of the Bessel functions is truncated to a power of eccentricity to reduce errors to within 10^{-11}. This is to prevent cumulative errors from exceeding the prescribed tolerance of 10^{-9}. Using $k = k_{\max} = 9$, $N_{\text{tol}} = 11$, and $e = 0.1$ in (35), the number of terms required in $J_9(9e)$ is $s = 1$. Consequently, the maximum power of eccentricity to which the approximate solution is developed, is $k_{\max} + 2s = 11$. Therefore, a completely analytical expansion for $\cos f$ within the desired tolerance, correct to $e = .1$, is given by:

$$\begin{aligned}
\cos f \approx -e + \frac{2\eta^2}{e}\Bigg[&\frac{e}{2}\cos M + \frac{e^2}{2}\cos 2M + \left(\frac{9}{16}\cos 3M - \frac{1}{16}\cos M\right)e^3 \\
&+ \left(\frac{2}{3}\cos 4M - \frac{1}{6}\cos 2M\right)e^4 + \left(\frac{625}{768}\cos 5M - \frac{81}{256}\cos 3M + \frac{1}{384}\cos M\right)e^5 \\
&+ \left(\frac{81}{80}\cos 6M - \frac{8}{15}\cos 4M + \frac{1}{48}\cos 2M\right)e^6 \\
&+ \left(\frac{117649}{92160}\cos 7M - \frac{15625}{18432}\cos 5M + \frac{729}{10240}\cos 3M - \frac{1}{18432}\cos M\right)e^7 \\
&+ \left(\frac{512}{315}\cos 8M - \frac{729}{560}\cos 6M + \frac{8}{45}\cos 4M - \frac{1}{720}\cos 2M\right)e^8 \\
&+ \left(\frac{4782969}{2293760}\cos 9M - \frac{5764801}{2949120}\cos 7M + \frac{390625}{1032192}\cos 5M - \frac{729}{81920}\cos 3M\right. \\
&\left.+\frac{1}{1474560}\cos M\right)e^9 + \left(-\frac{8192}{2835}\cos 8M + \frac{6561}{8960}\cos 6M - \frac{32}{945}\cos 4M\right. \\
&\left.+\frac{1}{17280}\cos 2M\right)e^{10} + \left(-\frac{387420489}{91750400}\cos 9M + \frac{282475249}{212336640}\cos 7M\right. \\
&\left.-\frac{9765625}{99090432}\cos 5M + \frac{6561}{9175040}\cos 3M - \frac{1}{176947200}\cos M\right)e^{11}\Bigg]
\end{aligned} \tag{39}$$

It is assumed that f is known and varies from 0 to 2π. The mean anomaly M corresponding to the true anomaly is obtained directly from Kepler's equation, via the eccentric anomaly E. The values for M are then used to calculate $\cos f$ using series solutions, and this is compared to the exact value. The errors due to truncation are shown in Figure 1. The solid line depicts the error if only $k_{\max} = 8$ orders of Bessel functions are used. In this case, the error is found to be greater than 10^{-9}, at $e = 0.1$. By using $k_{\max} = 9$, which was the value obtained from the approach described in this paper, the absolute value of the error is shown to be less than 10^{-9}, as shown by the dashed line. Furthermore, if the Bessel functions are expanded through $\mathcal{O}(e^{11})$, that is, if (39) is used, then the error, as shown by the dashed-dotted line, is of the same order.

As the second example, an expansion for $\sin f$, given by (21b), is sought within an accuracy of 10^{-6} and valid for $e \leq 0.6$. Using series b from (20b), with $p = 1$, $q = 2\eta = 1.6$, and $N_{\mathrm{tol}} = 6$, (27) and (28) both result in $k_{\max} = 44$. Using a Newton-Raphson iteration with an initial guess of $k_{\max} = 44$ on (37), results in $s = 18$, for the same eccentricity, but with a required tolerance of 10^{-9}. Consequently, the Bessel functions are to be evaluated through $\mathcal{O}(e^{78})$. The resulting terms are too numerous to be shown here, but can be easily generated using any symbolic algebra tool. The errors arising due to truncation are shown in Figure 2. The solid line depicts the error if the series is terminated at $k_{\max} = 43$. In this case, the error is slightly larger than the prescribed tolerance of 10^{-6}, and the use of one additional term ($k_{\max} = 44$) reduces the error magnitude, as shown by the dashed line. By restricting the maximum order of the eccentricity to $\mathcal{O}(e^{78})$, as shown by the dashed-dotted line, the error is nearly indistinguishable from the case where the Bessel functions are evaluated using standard library functions with standard floating point accuracy.

5 Conclusions

In this paper, a simple procedure has been developed that allows the truncation of previously-known infinite series representation of solutions to functions of Kepler's transcendental equation. For a given (maximum) eccentricity and numerical tolerance, the number of Bessel functions required is easily obtained using the Lambert W function. Furthermore, it is shown that the Bessel functions may themselves be terminated at an appropriate order of eccentricity and the use of infinite series in the function may be avoided. The final solution is therefore analytical for a given tolerance and known maximum eccentricity.

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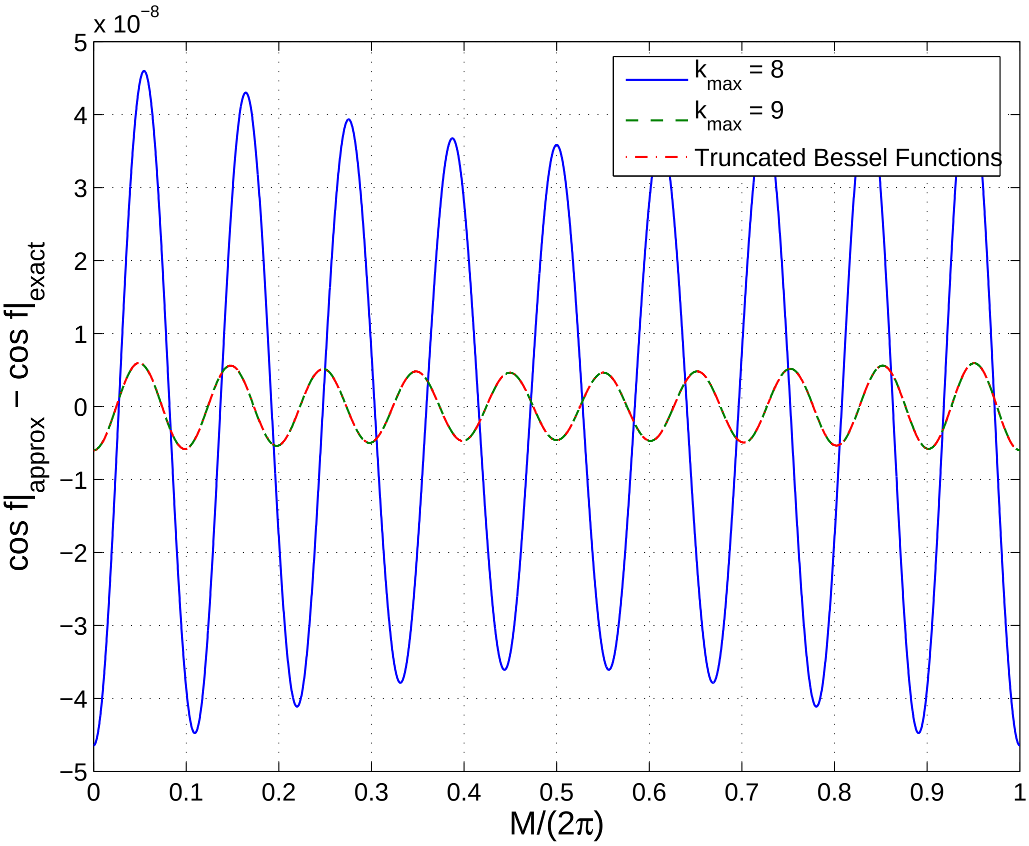

Fig. 1 Errors Between Approximate and Exact Functions for $\cos f$, Example 1, $N_{\mathrm{tol}} = 9$, $e = 0.1$

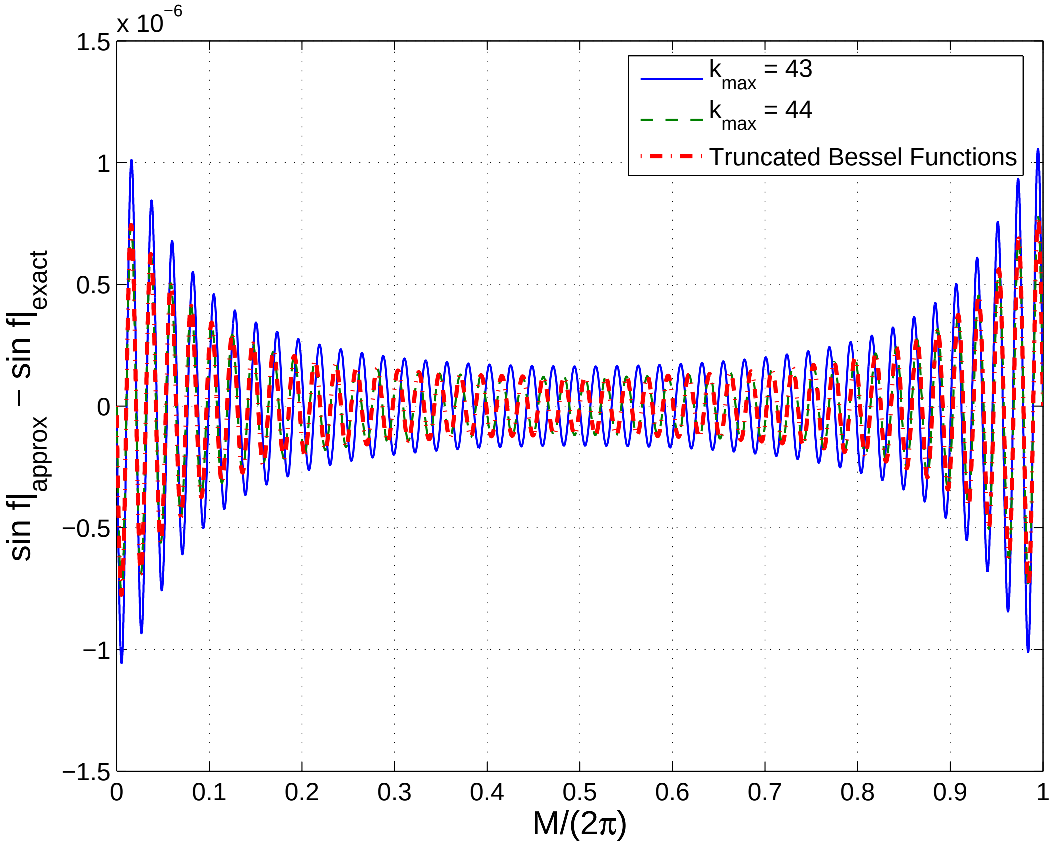

Fig. 2 Errors Between Approximate and Exact Functions for $\sin f$, Example 2, $N_{\text{tol}} = 6$, $e = 0.6$