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Prasenjit Sengupta · Srinivas R. Vadali ·

Kyle T. Alfriend

Second-Order State Transition for Relative Motion near Perturbed, Elliptic Orbits

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Abstract This paper develops a tensor and its inverse, for the analytical propagation of the position and velocity of a satellite, with respect to another, in an eccentric orbit. The tensor is useful for relative motion analysis where the separation distance between the two satellites is large. The use of nonsingular elements in the formulation ensures uniform validity even when the reference orbit is circular. Furthermore, when coupled with state transition matrices from existing works that account for perturbations due to Earth oblateness effects, its use can very accurately propagate relative states when oblateness effects and second-order nonlinearities from the differential gravitational field are of the same order of magnitude. The effectiveness of the tensor is illustrated with various examples.

Keywords formation flight · nonlinear relative motion · state transition · nonsingular elements

1 Introduction

The study of the relative motion problem requires the analysis of the motion of one satellite in a frame of reference attached to and moving with another satellite, for various purposes, such as on-orbit docking, and formation flight. The simplest model governing the dynamics of relative motion are the differential equations formulated by Hill (1878), and Clohessy and Wiltshire (1960), commonly known as the Hill-Clohessy-Wiltshire (HCW) model. These equations model relative motion close to

Department of Aerospace Engineering, TAMU 3141, Texas A&M University, College Station, TX 77843-3141, USA, E-mail: prasenjit@tamu.edu,svadali@aero.tamu.edu,alfriend@aero.tamu.edu

a circular orbit in a central gravitational field, thereby appearing as three, second-order, constant-coefficient, ordinary differential equations. A state transition matrix for this system, that can be used to linearly propagate the states, given the initial conditions, can easily be derived. Such a matrix is found useful in various problems, such as optimal control, and the solution to two-point boundary value problems, that require the sensitivity of a solution to a perturbation in initial conditions. However, the application of the STM based on the HCW equations is limited by the underlying assumptions of the model. The three most important factors that invalidate these assumptions are: eccentricity of the reference orbit, nonlinear differential gravity field (due to large relative distance), and perturbations in the gravitational field due to an oblate attracting body (Kaula 2000, chap. 1), the most significant of which arises due to the zonal harmonic coefficient J_2.

Relative motion equations and their analysis, when these assumptions are violated, have been extensively represented in the literature. The Tschauner-Hempel (TH) model (Tschauner and Hempel 1965) is also a linear model wherein relative position coordinates are scaled by the current radius of the reference orbit, and the true anomaly of the reference orbit is used as the independent variable, instead of time. The TH model is thus concisely written as three, second-order, linear equations with periodic coefficients, and is valid for all eccentricities of the reference orbit. The TH equations admit analytical solutions in terms of a special integral (Tschauner and Hempel 1965; Carter and Humi 1987; Carter 1990), also known as Lawden's integral (Lawden 1963, chap. 5). These solutions have been used to formulate state transition matrices (STMs) that are valid for arbitrary eccentricities, with the true anomaly as the independent variable, as shown by Wolfsberger et al. (1983), Carter (1998), and Yamanaka and Ankersen (2002). State transition matrices that use time as the independent variable have also been developed by Melton (2000) and Broucke (2003), although the former is limited to low eccentricities.

Relative motion is also completely analytically described by the use of orbital elements of each satellite. The geometric method (Alfriend et al. 2000) and the unit sphere model (Vadali 2002; Sengupta et al. 2004) are nonlinear models that require the orbital elements of both satellites to be propagated individually. Therefore, given the orbital elements of the reference satellite, once the initial orbital element differences corresponding to the initial relative states are known, the relative trajectory can be completely determined. However, the relationship between orbital element differences and relative states is nonlinear, and in general the former cannot be obtained from the latter in a straightforward manner. The linearization of these models, as shown by Alfriend et al. (2000), and also by Garrison et al. (1995), results in a matrix relationship between the relative states and differential orbital elements. Furthermore, relative motion can be defined using one independent variable, either time or any of the angular anomalies of the reference orbit. Thus, STMs based on this approach may also be derived.

The presence of Earth oblateness effects complicates the study of relative motion, by causing all the orbital elements of a satellite to change with time. These variations, to the first order in J_2 are classified as secular growth, short-periodic oscillations, and long-periodic oscillations. The J_2 potential introduces differential relative acceleration terms, and causes the rotating frame in which relative motion is analyzed, to precess. The complete nonlinear description of relative motion in the presence of J_2 perturbations has been developed by Kechichian (1998), although the system of equations presented therein cannot be solved in closed form. Simplified, linear relative motion models that include J_2 effects were developed by Vadali et al. (2002) and Schweigart and Sedwick (2002), but their use is limited to circular reference orbits and small relative orbits only. A STM was formulated by Gim and Alfriend (2003), using the geometric method in nonsingular orbital elements, and Brouwer theory (Brouwer 1959) to convert to and from differential mean and osculating elements. This STM can completely characterize linear relative motion in eccentric orbits. The use of nonsingular elements ensures uniform validity of the results for elliptic as well as circular orbits. A similar result was obtained by Yan et al. (2004), but by utilizing the unit sphere formulation for relative motion.

In spite of the extensive literature related to the rendezvous problem and formation flight, the effects of nonlinearity are yet to be modeled fully. The relative motion equations perturbed by higher-order differential gravity terms have been analyzed and solved with particular application to formation flight and periodic motion (Richardson and Mitchell 2003; Vaddi et al. 2003; Gurfil 2005b; Sengupta et al. 2006). By representing relative motion using spherical coordinates, and by the use of perturbation techniques, Karlgaard and Lutze (2003) solved the Clohessy-Wiltshire equations perturbed by second-order differential gravity terms with a circular reference. In their approach, a frequency correction was introduced to ensure uniform validity of the solution. Kasdin et al. (2005) used a Hamiltonian formulation to obtain solutions to a similar problem, but with J_2 perturbations, also with a circular orbit assumption. Yan (2006) presented expressions for J_2-perturbed relative motion coordinates, correct to the second order in J_2, by using the higher-order generating function from Brouwer theory. Consequently, small element differences up to the third order were also included, to ensure consistency between the magnitudes of second-order J_2, and nonlinearity effects. A map for relative position propagation using second partials from the unit-sphere approach, was also derived. However, the general case of nonlinear relative motion using state variables and one independent variable has not been solved.

This paper solves the relative motion problem to the second-order, by formulating a nonlinear map that can be used to find the relative states from differential orbital elements, that is accurate to the second order, and can therefore be used for large relative orbits. A method for solving the inverse problem is also devised, thereby leading to the formulation of a linear-quadratic map between the initial relative states, and the relative states at a desired time, using differential orbital elements as an intermediary. Furthermore, it is shown that the quadratic component of the map is sufficient for use with STMs that take into account J_2 perturbations. This is because, for many cases, quadratic terms and terms with J_2 are of the same order, and for large relative orbits, nonlinearity effects dominate. A nonsingular element set is used to ensure uniform validity even when the eccentricity of the reference orbit is small or zero.

2 Nonlinear Relative Motion

Consider an Earth-centered inertial (ECI) frame, denoted by $\mathcal{N}$, with orthonormal basis $\mathcal{B}_N = \{\mathbf{i}_x \; \mathbf{i}_y \; \mathbf{i}_z\}$. The vectors $\mathbf{i}_x$ and $\mathbf{i}_y$ lie in the equatorial plane, with $\mathbf{i}_x$ coinciding with the line of the equinoxes, and $\mathbf{i}_z$ passing through the North Pole. The vector $\mathbf{i}_y$ completes the triad by the relation $\mathbf{i}_y = \mathbf{i}_z \times \mathbf{i}_x$. Relative motion is conveniently described in a Local-Vertical-Local-Horizontal (LVLH) frame, as shown in Figure 1 and denoted by $\mathcal{L}$, that is attached to the target satellite (also called the leader or chief). This frame has basis $\mathcal{B}_L = \{\mathbf{i}_r \; \mathbf{i}_\theta \; \mathbf{i}_h\}$, with $\mathbf{i}_r$ lying along the radius vector from the Earth's center to the satellite, $\mathbf{i}_h$ coinciding with the normal to the orbital plane defined by the position and velocity vectors of the satellite, and $\mathbf{i}_\theta = \mathbf{i}_h \times \mathbf{i}_r$. Let the position of the deputy satellite (also called the follower or chaser) in the chief's LVLH frame be denoted by $\boldsymbol{\varrho} = \overline{x}\mathbf{i}_r + \overline{y}\mathbf{i}_\theta + \overline{z}\mathbf{i}_h$, where $\overline{x}$, $\overline{y}$ and $\overline{z}$ denote the components of the position vector along the radial, along-track, and out-of-plane directions, respectively. Assuming a central gravity field, the frame $\mathcal{L}$ rotates with angular velocity $\dot{\theta}\mathbf{i}_h$, where θ is the true argument of latitude, or the angle between the radius vector of the chief, and the nodal crossing. The quantity θ is given by the sum $\omega + f$, where ω is the argument of periapsis, and f is the true anomaly. Under the assumption of a central gravity field, ω is constant, and $\dot{\theta} = \dot{f}$.

In the rotating frame, the system of equations using relative position and velocity variables, is given by:

$$\ddot{\overline{x}} - 2\dot{\theta}\,\dot{\overline{y}} - \dot{\theta}^2\,\overline{x} - \ddot{\theta}\,\overline{y} = -\frac{\mu(r+\overline{x})}{\left[(r+\overline{x})^2+\overline{y}^2+\overline{z}^2\right]^{3/2}} + \frac{\mu}{r^2} \tag{1a}$$

$$\ddot{\overline{y}} + 2\dot{\theta}\,\dot{\overline{x}} - \dot{\theta}^2\,\overline{y} + \ddot{\theta}\,\overline{x} = -\frac{\mu\overline{y}}{\left[(r+\overline{x})^2+\overline{y}^2+\overline{z}^2\right]^{3/2}} \tag{1b}$$

$$\ddot{\overline{z}} = -\frac{\mu\overline{z}}{\left[(r+\overline{x})^2+\overline{y}^2+\overline{z}^2\right]^{3/2}} \tag{1c}$$

In the above equations, $\mu = GM_{\oplus}$, where G is the universal gravitation constant, and $M_{\oplus}$ is the mass of the central body, in this case, the Earth. Furthermore, r is the radial distance of the chief, given by $r = p/(1 + e\cos f)$, where $p = a\eta^2$ is the semiparameter, a is the semimajor axis, $\eta = \sqrt{(1 - e^2)}$, and e is the eccentricity of the chief's orbit. The symbols $(\dot{\ })$ and $(\ddot{\ })$ denote the first and second derivative with respect to time. These equations are nonlinear, and nonautonomous, with periodic coefficients that vary implicitly with time. Following the approach in Tschauner and Hempel (1965), the position is normalized with respect to the radius of the chief, and true anomaly is selected as the independent variable, where the formula $\dot{f} = h/r^2$, with specific angular momentum, $h = \sqrt{(\mu p)}$ is used. The normalized position, velocity, and acceleration vectors are then given as follows:

$$\boldsymbol{\rho} = x\mathbf{i}_r + y\mathbf{i}_\theta + z\mathbf{i}_h = (1 + e\cos f)\frac{\boldsymbol{\varrho}}{p} \tag{2a}$$

$$\boldsymbol{\rho}' = (1 + e\cos f)\frac{\boldsymbol{\varrho}'}{p} - e\sin f\,\frac{\boldsymbol{\varrho}}{p} \tag{2b}$$

$$\boldsymbol{\rho}'' = (1 + e\cos f)\frac{\boldsymbol{\varrho}''}{p} - 2e\sin f\,\frac{\boldsymbol{\varrho}'}{p} - e\cos f\,\frac{\boldsymbol{\varrho}}{p} \tag{2c}$$

where x, y, and z are the components of the normalized relative position, and $(')$ and $('')$ denote derivatives with respect to f, such that:

$$(\dot{\ }) = \dot{f}(\ '\ ) = \sqrt{\frac{\mu}{p^3}}(1 + e\cos f)^2(\ '\ ) \tag{3a}$$

$$
\begin{aligned}
(\ddot{\ }) &= \dot{f}^2(\ ''\ ) + \ddot{f}(\ '\ ) \\
&= \frac{\mu}{p^3}(1+e\cos f)^3\Big[(1+e\cos f)(\ ''\ ) - 2e\sin f(\ '\ )\Big]
\end{aligned}
\tag{3b}
$$

Using the scaled states and the new independent variable, it can be shown that (1) reduces to the following form:

$$
x'' = 2y' + \frac{1+x}{(1+e\cos f)}\left(1 - \frac{1}{d^3}\right) \tag{4a}
$$

$$
y'' = -2x' + \frac{y}{(1+e\cos f)}\left(1 - \frac{1}{d^3}\right) \tag{4b}
$$

$$
z'' = -\frac{z}{(1+e\cos f)}\left(e\cos f + \frac{1}{d^3}\right) \tag{4c}
$$

where $d = [(1+x)^2 + y^2 + z^2]^{1/2}$. Since $x, y, z \ll 1$, the term $1/d^3$ can be expanded as a series of Legendre polynomials, as shown by Sengupta et al. (2006), to yield the following system of equations:

$$
\begin{aligned}
&\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} + \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} + \begin{bmatrix} -3/(1+e\cos f) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \\
&= \sum_{k=3}^{\infty} \frac{(-1)^k k}{(1+e\cos f)(y^2+z^2)} \left[\rho^k P_k(x/\rho) \begin{Bmatrix} 0 \\ y \\ z \end{Bmatrix} + \rho^{k-1} P_{k-1}(x/\rho) \begin{Bmatrix} y^2+z^2 \\ -xy \\ -xz \end{Bmatrix}\right] \\
&= \frac{3}{2}(1+e\cos f)^{-1} \begin{Bmatrix} y^2+z^2-2x^2 \\ 2xy \\ 2xz \end{Bmatrix} + \mathcal{O}\left(|\boldsymbol{\rho}|^3\right)
\end{aligned}
\tag{5}
$$

where P_k is the kth Legendre polynomial. If second- and higher-order terms are neglected in the above system of equations, the TH equations are recovered, and their solution is known, as given by Lawden (1963); Tschauner and Hempel (1965); Carter and Humi (1987). The system of equations restricted to quadratic terms were presented by Euler and Shulman (1967). For the special case of periodic relative motion, this system can be solved using a straightforward expansion (Sengupta et al. 2006), without the need of a frequency correction because both the linear system and completely nonlinear system are 2π-periodic. However, for the more general case where periodicity conditions have not been enforced, it is not yet known whether a method using a frequency correction can successfully be applied. A formulation for relative motion equations with a circular reference orbit has been presented by Bond (1999), where the only assumption on the chaser orbit is that its eccentricity is low. Using the methodology described therein, it can be shown that all solutions are stable and bounded, which is true for any two satellites orbiting around a central body. The local in-plane growth term, for the simplest case of relative motion between two circular orbits, is given by $\sin(\Delta n\, t)$, where Δn is the difference between the mean motions of the chaser and target. This term can be expanded using a Taylor series comprising terms up to an arbitrary order:

$$\sin(\Delta n\, t) = \Delta n\, t - \frac{1}{6}(\Delta n\, t)^3 + \dots \tag{6}$$

Since terms comprising t^k, $k = 1 \dots \infty$ arise from the expansion process, standard perturbation methods cannot be used successfully to obtain a uniformly valid solution (Alfriend et al. 2002).

To circumvent the problems associated with an analytical solution to the perturbed system, the geometric method is used. Each orbit is uniquely defined by a set of nonsingular orbital elements, $\text{œ} = \{a\ \theta\ i\ q_1\ q_2\ \Omega\}^\top$, where i is the inclination, and Ω is the Right Ascension of the Ascending Node (RAAN). The quantities q_1 and q_2 are useful when eccentricity is near-zero, and are given by $q_1 + \jmath\, q_2 = e \exp(\jmath\, \omega)$. where $\jmath = \sqrt{-1}$.

In the geometric description, the relative position is given by the following expression:

$$\boldsymbol{\varrho} = \mathbf{C}_C \mathbf{C}_D^\top \mathbf{r}_D - \mathbf{r}_C \tag{7}$$

where $\mathbf{r}_D$ and $\mathbf{r}_C$ are the position vectors of the deputy and chief in their respective LVLH frames, and $\mathbf{C}_D$ and $\mathbf{C}_C$ are the direction cosine matrices transforming the components of a vector in the ECI frame, to the equivalent vector in the LVLH frames of the deputy and chief, respectively. The directions cosine matrix comprises the 3-1-3 Euler angles of RAAN, inclination, and argument of latitude. That is, $\mathbf{C}_C \equiv \mathbf{C}(\Omega_C, i_C, \theta_C)$ and $\mathbf{C}_D \equiv \mathbf{C}(\Omega_D, i_D, \theta_D)$. The components of the matrix $\mathbf{C}$ in terms of Ω, i, and θ, are given in Junkins and Turner (1986, chap. 2). If it is assumed that the orbital elements of the deputy are not much different from those of the chief, then the direction cosine matrix of the deputy can be expanded in a Taylor series, using the chief as a reference. For example, the differential inclination $\delta i = i_D - i_C$, if assumed small, allows expansions of the type $\cos \delta i \approx 1 - \delta i^2/2$ and $\sin \delta i \approx \delta i$. Using these approximations, and henceforth dropping the subscript 'C' for the chief's elements, the relative position is given by:

$$\begin{aligned}\overline{x} &\approx \delta r - \frac{r}{2}\left(\cos^2\theta + \cos^2 i\,\sin^2\theta\right)\delta\Omega^2 - \frac{r}{2}\sin^2\theta\,\delta i^2 - \frac{r}{2}\,\delta\theta^2 \\ &\quad + r\sin i\sin\theta\cos\theta\,\delta\Omega\,\delta i - r\cos i\,\delta\Omega\,\delta\theta \qquad (8a)\\ \overline{y} &\approx (r+\delta r)\left(\delta\theta + \cos i\,\delta\Omega\right) + \frac{r}{2}\sin^2 i\sin\theta\cos\theta\,\delta\Omega^2 - \frac{r}{2}\sin\theta\cos\theta\,\delta i^2 \\ &\quad - r\sin i\sin^2\theta\,\delta\Omega\,\delta i \qquad (8b)\\ \overline{z} &\approx (r+\delta r)\left(\sin\theta\,\delta i - \sin i\cos\theta\,\delta\Omega\right) + \frac{r}{2}\sin i\cos i\sin\theta\,\delta\Omega^2 \\ &\quad + r\sin i\sin\theta\,\delta\Omega\,\delta\theta + r\cos\theta\,\delta i\,\delta\theta \qquad (8c)\end{aligned}$$

It should be noted that if only linear terms are considered in (8), then the out-of-plane dynamics depend only on differential RAAN, $\delta\Omega$, and differential inclination, δi, and are therefore uncoupled from the in-plane dynamics. The inclusion of second-order terms introduces coupling effects. To formulate expressions for relative position in terms of the differential orbital elements, the expansions of the differential radius and differential true argument of latitude, are required. The first expansion can be obtained from the formula for the current radius of the satellite, which, in terms of a, q_1, q_2, and θ, is given by:

$$r = \frac{a(1 - q_1^2 - q_2^2)}{(1 + q_1\cos\theta + q_2\sin\theta)} \qquad (9)$$

For the sake of brevity, the following two functions are used throughout the paper:

$$\alpha = 1 + q_1\cos\theta + q_2\sin\theta \qquad (10a)$$

$$\beta = q_1\sin\theta - q_2\cos\theta \qquad (10b)$$

Using these functions, it is easily shown that:

$$\frac{\partial\alpha}{\partial\theta} = -\beta, \quad \frac{\partial\alpha}{\partial q_1} = \cos\theta, \quad \frac{\partial\alpha}{\partial q_2} = \sin\theta \qquad (11a)$$

$$\frac{\partial\beta}{\partial\theta} = \alpha - 1, \quad \frac{\partial\beta}{\partial q_1} = \sin\theta, \quad \frac{\partial\beta}{\partial q_2} = -\cos\theta \qquad (11b)$$

Using (10) and (11) to calculate the partials of r with respect to the orbital elements, the differential radius δr is obtained using a Taylor series expansion to the second order, in the differential elements

δa, δq_1, δq_2, and $\delta\theta$:

$$\begin{aligned}
\frac{\delta r}{r} \approx\ & \frac{\delta a}{a} + \frac{\beta}{\alpha}\,\delta\theta - \frac{1}{\eta^2\alpha}\left(2q_1\alpha + \eta^2\cos\theta\right)\,\delta q_1 - \frac{1}{\eta^2\alpha}\left(2q_2\alpha + \eta^2\sin\theta\right)\,\delta q_2 \\
& - \frac{1}{2\alpha^2}(\alpha^2 - 3\alpha + 2\eta^2)\,\delta\theta^2 + \frac{1}{\eta^2\alpha^2}\left(\eta^2\cos^2\theta + 2q_1\alpha\cos\theta - \alpha^2\right)\,\delta q_1^2 \\
& + \frac{1}{\eta^2\alpha^2}\left(\eta^2\sin^2\theta + 2q_2\alpha\sin\theta - \alpha^2\right)\,\delta q_2^2 + \frac{\beta}{\alpha}\,\frac{\delta a}{a}\,\delta\theta - \frac{1}{\eta^2\alpha}\left(2q_1\alpha + \eta^2\cos\theta\right)\,\frac{\delta a}{a}\,\delta q_1 \\
& - \frac{1}{\eta^2\alpha}\left(2q_2\alpha + \eta^2\sin\theta\right)\,\frac{\delta a}{a}\,\delta q_2 - \frac{1}{\eta^2\alpha^2}\left(2q_1\alpha\beta + 2\eta^2\beta\cos\theta - \eta^2\alpha\sin\theta\right)\,\delta\theta\,\delta q_1 \\
& - \frac{1}{\eta^2\alpha^2}\left(2q_2\alpha\beta + 2\eta^2\beta\sin\theta + \eta^2\alpha\cos\theta\right)\,\delta\theta\,\delta q_2 \\
& + \frac{2}{\eta^2\alpha^2}\left(q_1\alpha\sin\theta + q_2\alpha\cos\theta + \eta^2\sin\theta\cos\theta\right)\,\delta q_1\,\delta q_2
\end{aligned} \tag{12}$$

The second expression that is required, is the value of the differential argument of latitude, $\delta\theta$, in terms of its initial value. Since $\theta = \omega + f$, it follows that $\delta\theta = \delta\omega + \delta f$. By the use of the following formulae (Battin 1999, chap. 5):

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{13a}$$

$$M = E - e\sin E \tag{13b}$$

the differential true anomaly, correct to the second order, is given by:

$$\delta f = d_1\,\delta M + d_2\,\delta e + d_3\,\delta M^2 + d_4\,\delta e^2 + d_5\,\delta M\,\delta e \tag{14}$$

where,

$$d_1 = \frac{1}{\eta^3}(1 + e\cos f)^2 \tag{15a}$$

$$d_2 = \frac{1}{\eta^2}\sin f\,(2 + e\cos f) \tag{15b}$$

$$d_3 = -\frac{e}{\eta^6}\sin f\,(1 + e\cos f)^3 \tag{15c}$$

$$d_4 = \frac{1}{2\eta^4}\sin f\,(2e + 5\cos f + 6e\cos^2 f + 2e^2\cos^3 f) \tag{15d}$$

$$d_5 = -\frac{1}{\eta^5}(e - 2\cos f - 2e\cos^2 f)(1 + e\cos f)^2 \tag{15e}$$

Use is made of the mean argument of latitude, $\lambda = \omega + M$, which allows one to use the relation $\delta M = \delta\lambda - \delta\omega$. The following equations are also required to obtain $\delta\theta$ in terms of nonsingular elements:

$$e^2\,\delta\omega \approx -e\sin\omega\,\delta q_1 + e\cos\omega\,\delta q_2 + \sin\omega\,\cos\omega\left(\delta q_1^2 - \delta q_2^2\right) - \cos 2\omega\,\delta q_1\,\delta q_2 \tag{16a}$$

$$e\,\delta e \approx e\cos\omega\,\delta q_1 + e\sin\omega\,\delta q_2 + \frac{1}{2}\sin^2\omega\,\delta q_1^2 + \frac{1}{2}\cos^2\omega\,\delta q_2^2 - \frac{1}{2}\sin 2\omega\,\delta q_1\,\delta q_2 \tag{16b}$$

$$\delta\lambda = \delta\lambda_0 + \delta n(t - t_0) = \delta\lambda_0 + \left(-\frac{3}{2}\frac{\delta a}{a} + \frac{15}{8}\frac{\delta a^2}{a^2}\right) n(t - t_0) \tag{16c}$$

where $\delta\lambda_0$ is the initial mean argument of latitude difference. Kepler's equation using nonsingular elements is used to denote elapsed time:

$$\begin{aligned} K(\theta_2, \theta_1) &\triangleq \int_{\theta_1}^{\theta_2} \frac{\eta^3}{(1 + q_1\cos\theta + q_2\sin\theta)^2}\, d\theta \\ &= (F_2 - q_1\sin F_2 + q_2\cos F_2) - (F_1 - q_1\sin F_1 + q_2\sin F_1) \\ &= \lambda_2 - \lambda_1 = \dot{\lambda}\,\mathit{\Delta} t \end{aligned} \tag{17}$$

where $F = \omega + E$ is the eccentric argument of latitude, and $\dot{\lambda} = \dot{\omega} + \dot{M} = n$, for the two-body problem. The following equations relate the true and eccentric arguments of latitude:

$$\cos F = \frac{(1 + \eta - q_2^2)\cos\theta + q_1 q_2 \sin\theta + (1 + \eta)q_1}{(1 + \eta)(1 + q_1\cos\theta + q_2\sin\theta)} \tag{18a}$$

$$\sin F = \frac{(1 + \eta - q_1^2)\sin\theta + q_1 q_2 \cos\theta + (1 + \eta)q_2}{(1 + \eta)(1 + q_1\cos\theta + q_2\sin\theta)} \tag{18b}$$

$$\cos\theta = \frac{(1 + \eta - q_2^2)\cos F + q_1 q_2 \sin F - (1 + \eta)q_1}{(1 + \eta)(1 - q_1\cos F - q_2\sin F)} \tag{18c}$$

$$\sin\theta = \frac{(1 + \eta - q_1^2)\sin F + q_1 q_2 \cos F - (1 + \eta)q_2}{(1 + \eta)(1 - q_1\cos F - q_2\sin F)} \tag{18d}$$

Let $\boldsymbol{\delta œ} \triangleq \{\delta a/a\ \delta\theta\ \delta i\ \delta q_1\ \delta q_2\ \delta\Omega\}^\top$ denote the vector of differential orbital elements. The differential semimajor axis has been scaled by the reference semimajor axis to make it dimensionally equivalent to the other differential orbital elements. In the two-body problem, all orbital element differences are constant, except $\delta\theta$. Equation (14) and $\delta\omega$ are used to compose $\delta\theta$, and (15) and (16) are used to write $\delta\theta$ in terms of the nonsingular element differences, δq_1, δq_2, and $\delta\lambda$. The quantity $\delta\lambda$ is rewritten in terms of its initial value, $\delta\lambda_0$, which can be obtained from $\delta\theta_0$, using the inverse of the process described above. Consequently, the propagation of the differential orbital elements can be written in the following form:

$$\boldsymbol{\delta œ}(\theta_2) \approx \mathbf{G}(\theta_2, \theta_1)\, \boldsymbol{\delta œ}(\theta_1) + \frac{1}{2}\mathbf{H}(\theta_2, \theta_1) \otimes \boldsymbol{\delta œ}(\theta_1) \otimes \boldsymbol{\delta œ}(\theta_1) \tag{19}$$

where $\mathbf{G} \in \mathbb{R}^{6\times 6}$ is the transition matrix for the differential orbital elements, and $\mathbf{H} \in \mathbb{R}^{6\times 6\times 6}$ is the next-order transition tensor. In (19), the operator $\otimes$ denotes the dyadic product, i.e., in indicial notation this equation is equivalent to:

$$\delta œ_i(\theta_2) = G_{ij}(\theta_2, \theta_1)\, \delta œ_j(\theta_1) + \frac{1}{2}H_{ijk}(\theta_2, \theta_1)\, \delta œ_j(\theta_1)\, \delta œ_k(\theta_1) \tag{20}$$

with repeated indices implying summation. The components of $\mathbf{G}$ and $\mathbf{H}$ are given in Appendix A.

The expressions for relative velocities require the formulation of the quantities $\delta\dot{\theta}$ and $\delta\dot{r}$. These quantities can be calculated by taking the variations upto the second order in $\dot{\theta} = \dot{f} = \sqrt{\mu/p^3}\,\alpha^2$, and $\dot{r} = \sqrt{\mu/p}\,\beta$.

3 Formulation of the State Transition Matrix and Tensor

Let $\mathbf{x} = \{x\ y\ z\ x'\ y'\ z'\}^\top$ denote the state vector. The nondimensional relative velocities are given by $\boldsymbol{\rho}' = d\boldsymbol{\rho}/d\theta$. In the two-body problem, θ and f differ by a constant, ω; therefore a derivative with respect to θ is equivalent to a derivative with respect to f.

The objective is to obtain a mapping between the state vector at $\theta = \theta_2$, and the state vector at $\theta = \theta_1$, that includes quadratic nonlinearities. Consequently, an expression of the following type is required:

$$\mathbf{x}(\theta_2) = \mathbf{\Phi}^{(1)}(\theta_2, \theta_1)\,\mathbf{x}(\theta_1) + \frac{1}{2}\mathbf{\Phi}^{(2)}(\theta_2, \theta_1) \otimes \mathbf{x}(\theta_1) \otimes \mathbf{x}(\theta_1) \tag{21}$$

In this equation, $\mathbf{\Phi}^{(1)} \in \mathbb{R}^{6\times 6}$ is the STM for linearized relative motion. The second term contains $\mathbf{\Phi}^{(2)} \in \mathbb{R}^{6\times 6\times 6}$, which is a third-order tensor for state transition, accounting for second-order nonlinearities. Since a matrix is also a second-order tensor, both $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$ will be referred to as state-transition tensors (STTs).

The evaluation of the STTs is performed via the use of the mapping between the Cartesian LVLH states and differential orbital elements. Using (12) in (8), and after some algebra, the following mapping between $\mathbf{x}(\theta)$ and $\boldsymbol{\delta}\text{œ}(\theta)$ is obtained:

$$\mathbf{x}(\theta) = \mathbf{P}(\theta)\boldsymbol{\delta}\text{œ}(\theta) + \frac{1}{2}\mathbf{Q}(\theta) \otimes \boldsymbol{\delta}\text{œ}(\theta) \otimes \boldsymbol{\delta}\text{œ}(\theta) + \mathcal{O}\left(|\boldsymbol{\delta}\text{œ}(\theta)|^3\right) \tag{22}$$

where $\mathbf{P} \in \mathbb{R}^{6\times 6}$ and $\mathbf{Q} \in \mathbb{R}^{6\times 6\times 6}$ are provided in Appendix B. To formulate the expressions for $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$, an inverse map from $\mathbf{x}(\theta)$ to $\boldsymbol{\delta}\text{œ}(\theta)$ is required. Therefore, the objective is to solve for $\boldsymbol{\delta}\text{œ}$ in terms of the given state vector, $\mathbf{x}$, at any value of θ.

3.1 Inverse Map from States to Differential Orbital Elements

It is stated at the outset, that an inverse map for the linear system has been provided by Gim and Alfriend (2003). However, for better accuracy, an inverse map that is quadratic in the states, is sought. From the *vis-viva* integral (Battin 1999, chap. 3), the relationship between a, orbit energy, $\mathcal{E}$, and the velocity and position of a satellite, is given by:

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \tag{23}$$

Sengupta et al. (2006) have shown that by using position and velocity scaled in the sense of the TH equations, and with true anomaly as the independent variable, the energy difference between two neighboring orbits can be written in the following form:

$$\begin{aligned}\frac{\eta^2}{2}\frac{\delta\mathcal{E}}{\mathcal{E}} &= (2 + 3e\cos f + e^2)\, x + e\sin f\,(1 + e\cos f)\, x' + (1 + e\cos f)^2\, y' \\ &+\frac{1}{2}\Big[-(1-e^2)\, x^2 + (2 + 3e\cos f + e^2)\, y^2 + (1 + e\cos f + e^2\sin^2 f)\, z^2 \\ &+(1+e\cos f)^2(x'^2 + y'^2 + z'^2) + 2e\sin f\,(1+e\cos f)(xx' + yy' + zz') \\ &+2(1+e\cos f)^2(xy' - yx')\Big] + \mathcal{O}\left(|\boldsymbol{\rho}|^3\right)\end{aligned} \tag{24}$$

where $\mathcal{E}$ is used to denote the orbital energy of the chief. The third- and higher-order terms in (24) are contributed by Legendre polynomials of equivalent order, and contribute relative position terms only. From (23), the semimajor axis difference can also be written in terms of the energy difference. To the second order, the equation relating the two quantities is as follows:

$$\delta a = \frac{\mu}{2\mathcal{E}^2}\,\delta\mathcal{E} - \frac{\mu}{2\mathcal{E}^3}\,\delta\mathcal{E}^2 + \mathcal{O}(\delta\mathcal{E}^3) \tag{25}$$

$$\text{or,}\quad \frac{\delta a}{a} \approx -\frac{\delta\mathcal{E}}{\mathcal{E}} + \left(\frac{\delta\mathcal{E}}{\mathcal{E}}\right)^2 \tag{26}$$

Upon substituting (24) in (26), and keeping terms up to the second order, an expression for the semimajor axis difference is obtained, in terms of the relative position and velocity components.

However, the procedure for determining expressions for the remaining five differential orbital elements is far more complicated, because the *vis-viva* integral is the only constant of two-body motion that relates a single orbital element to the position and velocity variables of a satellite. The remaining integrals - the eccentricity and angular momentum vectors - yield equations in multiple orbital elements, and furthermore, these equations have to be rewritten in inertial coordinates to introduce inclination and RAAN.

A related approach is to first solve for the position and velocity of the deputy in the ECI frame, $\mathcal{N}$, given the relative states in $\mathcal{B}$, and then convert the ECI position and velocity to the orbital elements of the chief. Vadali et al. (2002) present a set of nonlinear equations that can be used to calculate relative position and velocity in $\mathcal{B}$, when the relative position and velocity in $\mathcal{N}$, and position and velocity of the chief in $\mathcal{N}$, are provided. Consequently, for the inverse operation, to obtain the position and velocity of the deputy in $\mathcal{N}$, a set of nonlinear equations in relative states is obtained. The second step of conversion from the deputy's ECI states to the deputy's orbital elements, further complicates the approach.

Both methods discussed in this section result in a set of nonlinear equations, which can be solved using numerical techniques, but are not amenable to closed-form analysis. A straightforward transformation, using a reversion of series, is the subject of the next section.

3.2 Inverse Map using Series Reversion

In this paper, a reversion of series is chosen as a feasible transformation, since $|\boldsymbol{\rho}| \ll 1$, second-order terms in $|\boldsymbol{\rho}|$ are much smaller in magnitude than first-order terms. The reversion of series for a function of one or many variables (Feagin and Gottlieb 1971), can also be written using tensors of increasing order (Turner 2003). In the present case, following the development in Turner (2003), a reversion of series is applied on (22), comprising linear and quadratic terms only. Series reversion leads to the following approximate solution for $\boldsymbol{\delta}\text{œ}(\theta)$, for a given state vector $\mathbf{x}(\theta)$, and θ:

$$\boldsymbol{\delta}\text{œ}(\theta) = \mathbf{P}^{-1}(\theta)\,\mathbf{x}(\theta) - \frac{1}{2}\mathbf{P}^{-1}(\theta)\,\left\{\mathbf{Q}(\theta) \otimes \left[\mathbf{P}^{-1}(\theta)\,\mathbf{x}(\theta)\right] \otimes \left[\mathbf{P}^{-1}(\theta)\,\mathbf{x}(\theta)\right]\right\} + \mathcal{O}\left(|\mathbf{x}(\theta)|^3\right) \tag{27}$$

Let $\mathbf{R}(\theta) = \mathbf{P}^{-1}(\theta)$. Furthermore, let $\mathbf{S}(\theta_2, \theta_1)$ be a tensor such that its operation on the vector $\mathbf{x}(\theta_1)$ is equivalent to the following:

$$\begin{aligned}\mathbf{S}(\theta_2,\theta_1)\otimes\mathbf{x}(\theta_1)\otimes\mathbf{x}(\theta_1) &= \mathbf{Q}(\theta_2)\otimes\left[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{P}^{-1}(\theta_1)\,\mathbf{x}(\theta_1)\right]\otimes\left[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{P}^{-1}(\theta_1)\,\mathbf{x}(\theta_1)\right] \\ &= \mathbf{Q}(\theta_2)\otimes[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1)]\otimes[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1)]\end{aligned} \tag{28}$$

From Appendix A, it is observed that when $\theta_2 = \theta_1$, $\mathbf{G}(\theta_2, \theta_1)$ is the identity matrix. The matrix $\mathbf{G}(\theta_2, \theta_1)$ is introduced to facilitate the formulation of the STT, as will be made clear later. The above equation can be written in indicial notation as follows:

$$\begin{aligned} S_{ijk}(\theta_2,\theta_1)\, x_j(\theta_1)\, x_k(\theta_1) &= Q_{ijk}(\theta_2)\, G_{jn}(\theta_2,\theta_1) R_{nl}(\theta_1)\, x_l(\theta_1)\, G_{ko}(\theta_2,\theta_1) R_{om}(\theta_1)\, x_m(\theta_1) \\ &= Q_{ilm}(\theta_2)\, G_{ln}(\theta_2,\theta_1) R_{nj}(\theta_1)\, G_{mo}(\theta_2,\theta_1) R_{ok}(\theta_1)\, x_j(\theta_1)\, x_k(\theta_1) \end{aligned} \tag{29}$$

In the above expression, j, k, l and m are repeated indices and can therefore be interchanged. It follows that:

$$S_{ijk}(\theta_2,\theta_1) = Q_{ilm}(\theta_2)\, G_{ln}(\theta_2,\theta_1) R_{nj}(\theta_1)\, G_{mo}(\theta_2,\theta_1) R_{ok}(\theta_1) \tag{30}$$

In matrix notation, this is rewritten as:

$$\mathbf{S}_i(\theta_2,\theta_1) = \left[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\right]^\top \mathbf{Q}_i(\theta_2)\left[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\right] \tag{31}$$

$$\text{where, } \mathbf{S}_i = \begin{bmatrix} S_{i11} & \cdots & S_{i16} \\ \vdots & & \vdots \\ S_{i61} & \cdots & S_{i66} \end{bmatrix}, \quad \mathbf{Q}_i = \begin{bmatrix} Q_{i11} & \cdots & Q_{i16} \\ \vdots & & \vdots \\ Q_{i61} & \cdots & Q_{i66} \end{bmatrix}$$

It is obvious that $S_{ijk} = S_{ikj}$. Upon substituting the tensor $\mathbf{S}(\theta_1,\theta_1)$ from (30) in (27), the following equation is obtained from which $\boldsymbol{\delta}\text{œ}$ can be calculated from $\mathbf{x}$ at $\theta = \theta_1$:

$$\boldsymbol{\delta}\text{œ}(\theta_1) = \mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1) - \frac{1}{2}\mathbf{R}(\theta_1)\left[\mathbf{S}(\theta_1,\theta_1)\otimes\mathbf{x}(\theta_1)\otimes\mathbf{x}(\theta_1)\right] + \mathcal{O}\left(|\mathbf{x}(\theta_1)|^3\right) \tag{32}$$

Substituting (32) and (19) in (22) evaluated at $\theta = \theta_2$ leads to the following expression:

$$\begin{aligned} \mathbf{x}(\theta_2) &= \mathbf{P}(\theta_2)\,\mathbf{G}(\theta_2,\theta_1)\,\boldsymbol{\delta}\text{œ}(\theta_1) + \frac{1}{2}\mathbf{P}(\theta_2)\,\mathbf{H}(\theta_2,\theta_1)\otimes\boldsymbol{\delta}\text{œ}(\theta_1)\otimes\boldsymbol{\delta}\text{œ}(\theta_1) \\ &\quad + \frac{1}{2}\mathbf{Q}(\theta_2)\otimes\left[\mathbf{G}(\theta_2,\theta_1)\,\boldsymbol{\delta}\text{œ}(\theta_1)\right]\otimes\left[\mathbf{G}(\theta_2,\theta_1)\,\boldsymbol{\delta}\text{œ}(\theta_1)\right] + \mathcal{O}\left(|\boldsymbol{\delta}\text{œ}(\theta_1)|^3\right) \qquad (33) \\ &= \mathbf{P}(\theta_2)\,\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1) - \frac{1}{2}\mathbf{P}(\theta_2)\,\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\left[\mathbf{S}(\theta_1,\theta_1)\otimes\mathbf{x}(\theta_1)\otimes\mathbf{x}(\theta_1)\right] \\ &\quad + \frac{1}{2}\mathbf{P}(\theta_2)\,\mathbf{H}(\theta_2,\theta_1)\otimes\left[\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1)\right]\otimes\left[\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1)\right] \\ &\quad + \frac{1}{2}\mathbf{Q}(\theta_2)\otimes\left[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1)\right]\otimes\left[\mathbf{G}(\theta_2,\theta_1)\,\mathbf{R}(\theta_1)\,\mathbf{x}(\theta_1)\right] + \mathcal{O}(|\mathbf{x}(\theta_1)|^3) \qquad (34) \end{aligned}$$

Upon comparing (34) with (21), and by using (30), the following expressions for the STTs are obtained:

$$\Phi^{(1)}_{ij}(\theta_2,\theta_1) = P_{ik}(\theta_2)\, G_{kl}(\theta_2,\theta_1)\, R_{lj}(\theta_1) \tag{35a}$$

$$\Phi^{(2)}_{ijk}(\theta_2,\theta_1) = S_{ijk}(\theta_2,\theta_1) - P_{in}(\theta_2)\, G_{nm}(\theta_2,\theta_1)\, R_{ml}(\theta_1)\, S_{ljk}(\theta_1,\theta_1)$$
$$+P_{in}(\theta_2)\, H_{nlm}(\theta_2,\theta_1)\, R_{lj}(\theta_1)\, R_{mk}(\theta_1) \qquad (35b)$$

Thus, $\mathbf{x}(\theta_2)$ can be obtained in terms of $\mathbf{x}(\theta_1)$, if the matrices $\mathbf{P}$, $\mathbf{R}$, and tensor $\mathbf{Q}$ are known. Furthermore, let $\overline{\mathbf{S}}(\theta)$ denote a tensor such that $\overline{S}_{ijk}(\theta) = -R_{il}(\theta)\, S_{ljk}(\theta,\theta)$. Then, (32) reduces to the following form:

$$\boldsymbol{\delta}\boldsymbol{œ}(\theta) = \mathbf{R}(\theta)\,\mathbf{x}(\theta) + \frac{1}{2}\overline{\mathbf{S}}(\theta,\theta)\otimes\mathbf{x}(\theta)\otimes\mathbf{x}(\theta) + \mathcal{O}\left(|\mathbf{x}(\theta)|^3\right) \qquad (36)$$

The tensors $\mathbf{R}$ and $\overline{\mathbf{S}}$ are presented in Appendix B. It is worth noting that $\delta a/a$ obtained from (36) matches the expression obtained from (26). Though not shown in this paper, the properties of state transition matrices and tensors can easily be verified using indicial notation, by noting that $K(\theta_3,\theta_1) = K(\theta_3,\theta_2) + K(\theta_2,\theta_1)$, and $K(\theta_1,\theta_1) = 0$.

For the sake of completeness, the following relations can be used to transform between the dimensional and scaled states:

$$\begin{Bmatrix} \boldsymbol{\varrho} \\ \dot{\boldsymbol{\varrho}} \end{Bmatrix} = \begin{bmatrix} p/\alpha\,\mathbb{1}_3 & \mathbb{O}_3 \\ \sqrt{(\mu/p)}\,\beta,\mathbb{1}_3 & \sqrt{(\mu/p)}\,\alpha\,\mathbb{1}_3 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\rho}' \end{Bmatrix} \qquad (37a)$$

$$\begin{Bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\rho}' \end{Bmatrix} = \begin{bmatrix} (1/p)\,\alpha\,\mathbb{1}_3 & \mathbb{O}_3 \\ -(1/p)\,\beta\,\mathbb{1}_3 & \sqrt{(p/\mu)}/\alpha\,\mathbb{1}_3 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varrho} \\ \dot{\boldsymbol{\varrho}} \end{Bmatrix} \qquad (37b)$$

where $\mathbb{1}_m$ is the mth-order identity matrix, and $\mathbb{O}_m$ is the mth-order zero matrix. The matrices in (37) are denoted by $\mathbf{T}(\theta)$ and $\mathbf{T}^{-1}(\theta)$, respectively.

Since the analysis in this paper uses nonsingular orbital elements, the intermediate tensors used to formulate the STTs are still singular for equatorial or near-equatorial orbits ($i \sim 0$). This is easily observed from the expressions in Appendix B, some of which are composed of $\cot i$ and $\csc i$. This singularity may be avoided by using equinoctial elements (Broucke and Cefola 1972), or Euler parameters (Sengupta and Vadali 2005; Gurfil 2005a). Formulations based on these two parametrizations are more complicated and beyond the scope of this paper. A state transition matrix for the linear, J_2-perturbed case, that uses equinoctial elements, was developed by Gim and Alfriend (2005).

4 Oblateness Effects

The most significant advantage of the formulation in the previous section is that the second-order STT can very easily be used with more accurate first-order STTs that also account for J_2 perturbations.

It is first necessary to compare the order of the terms arising due to J_2 and nonlinear differential gravity. Sengupta et al. (2006) show that if the states of the TH equations are scaled by $\varrho_0/(1+\overline{e}\cos f)$, instead of $\overline{p}/(1+\overline{e}\cos f)$, where ϱ_0 represents the size of the relative orbit, then the second-order terms in x, y, and z have $\epsilon = \varrho_0/\overline{p}$ as a coefficient, where the overbar indicates the use of mean elements. Similarly, as shown by Kasdin et al. (2005), the linear terms that arise due to the inclusion of terms with J_2 in the gravitational potential, have $\overline{J} = J_2(R_\oplus/\overline{p})^2$ as their coefficient, where $R_\oplus$ is the radius of the Earth. Scaling the states by $\varrho_0/(1+\overline{e}\cos f)$ leads to $|\boldsymbol{\rho}| = \mathcal{O}(1)$, and consequently both linear and quadratic terms are of the same order, and the effect of J_2 and nonlinearity can be studied by comparing ϵ and $\overline{J}$. Assuming a reference orbit with $a = 13,000\,\mathrm{km}$, and $e = 0.3$, and using $J_2 = 1.08269\times10^{-3}$ and $R_\oplus = 6,378.14\,\mathrm{km}$, calculations show that $\overline{J} = 3\times10^{-4}$. For small relative orbits ($\varrho_0 < 500\,\mathrm{m}$), $\epsilon = 4.2\times10^{-5}$, and the inclusion of second-order terms may not lead to significant improvement in the accuracy of the propagated states, since oblateness effects dominate. However, second-order terms may be required when the chief's orbit is highly eccentric, even when the relative orbit is small, since the nonlinear terms also have $(1 + e\cos f)$ in the denominator of (5), and can be significantly dominant. In all cases of relative motion involving large relative orbits, the inclusion of second-order terms will improve the accuracy of the state transition of relative motion.

The above analysis also shows that the second-order terms can be limited to their representation in mean elements, and the inclusion of short-periodic and long-periodic variations are not necessary. If $\overline{œ}$ denotes a mean element, then the corresponding osculating element is given by $œ = \overline{œ}\,[1 + \mathcal{O}(J_2)]$. Consequently, $œ^2 = \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon J_2) + \mathcal{O}(J_2^2)$. Since the mean to osculating conversion employed in Gim and Alfriend (2003) is correct to first order in J_2, and $\epsilon J_2 = \mathcal{O}(J_2^2)$, only secular growth terms need to be incorporated in the higher-order terms. Therefore, in $\mathbf{Q}$, the mean elements $\overline{a}$, $\overline{e}$, and $\overline{i}$ are used instead of the osculating semimajor axis, eccentricity, and inclination. The RAAN, argument of periapsis, and mean anomaly, are propagated linearly with time, by using their mean secular rates, denoted by $\dot{\Omega}_s$, $\dot{\omega}_s$, and $\dot{M}_s$, respectively. In terms of nonsingular elements, the following equations are obtained (Gim and Alfriend 2003):

$$\overline{q}_1 = \overline{e}\cos(\overline{\omega}_0 + \dot{\omega}_s \Delta t) = \overline{q}_{10}\cos(\dot{\omega}_s\,\Delta t) - \overline{q}_{20}\sin(\dot{\omega}_s\,\Delta t) \tag{38a}$$

$$\overline{q}_2 = \overline{e}\sin(\overline{\omega}_0 + \dot{\omega}_s \Delta t) = \overline{q}_{10}\sin(\dot{\omega}_s\,\Delta t) + \overline{q}_{20}\cos(\dot{\omega}_s\,\Delta t) \tag{38b}$$

$$\overline{\lambda} = \overline{\omega} + \overline{M} = \overline{\lambda}_0 + (\dot{\omega}_s + \dot{M}_s)\,\Delta t \tag{38c}$$

where

$$\dot{\omega}_s = -\frac{3}{4}\,\overline{J}\,\overline{n}(1 - 5\cos^2\overline{i}) \tag{39a}$$

$$\dot{M}_s = \overline{n}\left[1 - \frac{3}{4}\,\overline{J}\eta\,(1 - 3\cos^2\overline{i}^2)\right] \tag{39b}$$

and $\overline{n} = \sqrt{(\mu/\overline{a}^3)}$. It should be noted that since the RAAN does not appear in of the elements of $\mathbf{Q}$, its secular rate is not required when calculating $\mathbf{Q}$.

For consistent analysis, notation is borrowed from Gim and Alfriend (2003), with the understanding that the true argument of latitude is the independent variable, instead of time. This is easily accommodated for, as the time can be obtained from the true argument of latitude without the inverse solution to Kepler's equation, by using (17) and (18). Additionally, the state vector in Gim and Alfriend (2003) is defined as $\{\overline{x}\ \dot{\overline{x}}\ \overline{y}\ \dot{\overline{y}}\ \overline{z}\ \dot{\overline{z}}\}^\top$; consequently it is necessary to use the matrix $\mathbf{T}$ as defined in (37) to transform between scaled and unscaled states, as well as to define the permutation matrix $\mathbf{\Pi}$, whose nonzero entries are:

$$\mathit{\Pi}_{11} = \mathit{\Pi}_{23} = \mathit{\Pi}_{35} = \mathit{\Pi}_{42} = \mathit{\Pi}_{54} = \mathit{\Pi}_{66} = 1 \tag{40}$$

Lastly, the differential orbital element set used in Gim and Alfriend (2003) uses the unscaled differential semimajor axis, δ. Therefore, a scaling matrix $\mathbf{\Gamma}(\text{œ})$ is also defined, whose nonzero entries are:

$$\mathit{\Gamma}_{11} = a,\ \mathit{\Gamma}_{22} = \mathit{\Gamma}_{33} = \mathit{\Gamma}_{44} = \mathit{\Gamma}_{55} = \mathit{\Gamma}_{66} = 1 \tag{41}$$

The inverse matrices $\mathbf{\Pi}^{-1}$ and $\mathbf{\Gamma}^{-1}$ are trivially obtained, by inspection.

Let $\mathbf{\Sigma}(\theta)$ be the first-order geometric map between the differential orbital elements, and the unscaled states, as defined by Gim and Alfriend (2003). It follows that the map between $\mathbf{x}(\theta)$ and $\boldsymbol{\delta}\text{œ}(\theta)$ is given by:

$$\mathbf{x}(\theta) = \mathbf{T}^{-1}(\theta)\,\mathbf{\Pi}\,\mathbf{\Sigma}(\theta)\,\mathbf{\Gamma}(\text{œ})\,\boldsymbol{\delta}\text{œ}(\theta) \tag{42}$$

The osculating differential nonsingular elements can be written in terms of the initial mean differential nonsingular elements, denoted by $\boldsymbol{\delta}\overline{\text{œ}}(\theta_1)$, by the following equation:

$$\boldsymbol{\Gamma}(\text{œ})\,\boldsymbol{\delta}\text{œ}(\theta_2) = \mathbf{D}(\theta_2)\,\overline{\boldsymbol{\Phi}}_{\overline{e}}(\theta_2,\theta_1)\,\boldsymbol{\Gamma}(\overline{\text{œ}})\,\boldsymbol{\delta}\overline{\text{œ}}(\theta_1) \tag{43}$$

where $\mathbf{D}(\theta) = \partial\boldsymbol{\delta}\text{œ}(\theta)/\partial\boldsymbol{\delta}\overline{\text{œ}}(\theta)$, and $\overline{\boldsymbol{\Phi}}_{\overline{e}}$ is the transition matrix mapping initial mean nonsingular element differences at θ_1, to the element differences at θ_2, as given by Gim and Alfriend (2003). In the absence of J_2 perturbations, $\overline{\boldsymbol{\Phi}}_{\overline{e}} = \mathbf{G}$, and $\boldsymbol{\Sigma} = \mathbf{P}$, as defined in the appendices.

To include second-order terms in the map from initial mean orbital element differences to the osculating relative states, it is first necessary to define $\boldsymbol{\Xi}$ as the following matrix:

$$\boldsymbol{\Xi} = \overline{\boldsymbol{\Gamma}}^{-1}\overline{\boldsymbol{\Phi}}_{\overline{e}}\overline{\boldsymbol{\Gamma}} \tag{44}$$

where $\overline{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}(\overline{\text{œ}})$. It can be shown that:

$$\varXi_{ij} = \begin{cases} \overline{\varPhi}_{\overline{e}_{ij}}, & i = 2\ldots 6,\ j = 2\ldots 6 \text{ and } i = j = 1 \\ \overline{\varPhi}_{\overline{e}_{ij}}/\overline{a}, & i = 1, j = 2\ldots 6 \\ \overline{a}\,\overline{\varPhi}_{\overline{e}_{ij}}, & j = 1, i = 2\ldots 6 \end{cases} \tag{45}$$

Appending the tensors $\overline{\mathbf{Q}}$ and $\overline{\mathbf{H}}$ to (42), where the overbar denotes mean elements, and using (44), results in the following map:

$$\begin{aligned}\mathbf{x}(\theta_2) &= \mathbf{T}^{-1}(\theta_2)\,\boldsymbol{\Pi}\,\boldsymbol{\Sigma}(\theta_2)\,\mathbf{D}(\theta_2)\,\overline{\boldsymbol{\Phi}}_{\overline{e}}(\theta_2,\theta_1)\,\overline{\boldsymbol{\Gamma}}\,\boldsymbol{\delta}\overline{\text{œ}}(\theta_1) + \frac{1}{2}\overline{\boldsymbol{\Sigma}}(\theta_2)\,\overline{\mathbf{H}}(\theta_2,\theta_1)\otimes\boldsymbol{\delta}\overline{\text{œ}}(\theta_1)\otimes\boldsymbol{\delta}\overline{\text{œ}}(\theta_1) \\ &\quad + \frac{1}{2}\overline{\mathbf{Q}}(\theta_2)\otimes[\boldsymbol{\Xi}\,\boldsymbol{\delta}\overline{\text{œ}}(\theta_1)]\otimes[\boldsymbol{\Xi}\,\boldsymbol{\delta}\overline{\text{œ}}(\theta_1)] \qquad (46) \\ &= \widetilde{\mathbf{P}}(\theta_2,\theta_1)\,\boldsymbol{\delta}\overline{\text{œ}}(\theta_1) + \frac{1}{2}\widetilde{\mathbf{Q}}(\theta_2,\theta_1)\otimes\boldsymbol{\delta}\overline{\text{œ}}(\theta_1)\otimes\boldsymbol{\delta}\overline{\text{œ}}(\theta_1) \qquad (47)\end{aligned}$$

where,

$$\widetilde{\mathbf{P}}(\theta_2,\theta_1) = \mathbf{T}^{-1}(\theta_2)\,\boldsymbol{\Pi}\,\boldsymbol{\Sigma}(\theta_2)\,\mathbf{D}(\theta_2)\,\overline{\boldsymbol{\Phi}}_{\overline{e}}(\theta_2,\theta_1)\,\overline{\boldsymbol{\Gamma}} \tag{48a}$$

$$\widetilde{Q}_{ijk}(\theta_2,\theta_1) = \overline{\varSigma}_{il}(\theta_2)\,\overline{H}_{ljk}(\theta_2,\theta_1) + \overline{Q}_{ilm}(\theta_2)\,\varXi_{lj}\,\varXi_{mk} \tag{48b}$$

In (44), it is necessary to use $\overline{\boldsymbol{\Phi}}_{\overline{e}}$, instead of $\overline{\mathbf{G}}$, even though $\boldsymbol{\Xi}$ appears at the quadratic level in (48*b*). This occurs because the structure of $\overline{\boldsymbol{\Phi}}_{\overline{e}}$ in Gim and Alfriend (2003) introduces terms of the order of $\overline{J}^2 K(\theta_2,\theta_1)^2/\overline{n}^2$, which can become $\mathcal{O}(\overline{J})$ after a few orbits.

The structure of (47) allows a straightforward application of a reversion of series on this equation, as outlined in the paper, by using $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ in place of $\mathbf{P}$ and $\mathbf{Q}$, in (32). It should be noted that for the initial conditions, $\overline{\boldsymbol{\Phi}}_{\overline{e}} = \mathbb{1}_6$ and $\overline{H}_{ijk} = 0$.

The propagation of the states using the second-order tensor does not require additional computation, since the matrices $\boldsymbol{\Sigma}$, $\mathbf{D}$, and $\overline{\boldsymbol{\Phi}}_{\overline{e}}$ are already available in the literature, and are required even in the linear case for the perturbed model.

5 Numerical Simulations

5.1 Validation of the Inverse Transformation from Relative States to Differential Orbital Elements (No Oblateness Effects)

The effectiveness of series reversion to obtain initial differential orbital elements, using $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ in (27) is tested in this section by way of an example. Consider a reference orbit with the following orbital elements:

$$\begin{aligned} a &= 13,000\,\mathrm{km}, \quad \theta_0 = 0.1\,\mathrm{rad}, \qquad i = 0.87266\,\mathrm{rad} \\ q_1 &= 0.29886, \qquad q_2 = 0.02615, \quad \Omega = 0.34907\,\mathrm{rad} \end{aligned} \tag{49}$$

Relative motion is established by selecting the following initial relative position and velocity in the LVLH frame:

$$\begin{aligned} \overline{x}_0 &= -3.0331\,\mathrm{km}, \; \overline{y}_0 = -12.967\,\mathrm{km}, \; \overline{z}_0 = 3.0837\,\mathrm{km} \\ \dot{\overline{x}}_0 &= -10.3931\,\mathrm{m/s}, \; \dot{\overline{y}}_0 = 4.3801\,\mathrm{m/s}, \; \dot{\overline{z}}_0 = 37.6743\,\mathrm{m/s} \end{aligned} \tag{50}$$

The relative position and velocity of the deputy are converted to the position and velocity in the ECI frame, using the procedure outlined by Vadali et al. (2002). These are used as initial conditions to integrate the ECI equations of motion in a central field without J_2 accelerations. At each instant, the ECI position and velocity of the chief and deputy are converted to their respective orbital elements, and the former is subtracted from the latter to provide the true differential orbital elements. The ECI positions and velocities of the two satellites are also converted to the relative position and velocity in the LVLH frame of the chief. This relative trajectory is shown in Figure 2, with the circle denoting epoch. The relative distance involved is more than 50 km, with a reference orbit eccentricity of 0.3.

For such orbits, it is not expected that linearized analysis for the inverse transformation, or state propagation, will yield satisfactory results.

The orbital elements are now calculated over a period of 10 orbits of the chief, using the inverse mapping derived, as shown in (32). The results are shown in Figures 3(a)-3(f). In all figures, the solid line indicates differential orbital elements obtained using only the linear part of the transformation, given by $\mathbf{R}(\theta)\,\mathbf{x}(\theta)$. The broken line shows the differential orbital elements obtained using the second-order map. The errors are shown on a logarithmic scale to bring out the accuracy of the second-order method. In all differential orbital elements, the second-order map reduces errors by three magnitudes on an average.

5.2 State Transition Including Oblateness Effects

To account for oblateness effects, the initial mean orbital elements of the chief's trajectory are selected as shown in (49). These are converted to osculating elements using Brouwer theory, and then to ECI position and velocity. Relative position and velocity of the deputy are selected as shown in (50). The integration of the ECI equations for the deputy and chief, includes the J_2 accelerations, and is performed for a time span corresponding to 10 orbits of the chief. The relative trajectory in this case is not expected to match Figure 2 due to the inclusion of J_2 effects; it is however, sufficient to note that the relative orbit size is of the same order as that in the non-J_2 case.

The initial mean orbital element differences are calculated using series reversion on (47). This reversion is straightforward since $\theta_2 = \theta_1$, and the inverse of the matrices are known. From the linear transformation, the mean differential orbital elements, denoted by $\boldsymbol{\delta}\overline{\text{œ}}_1$, are obtained as:

$$\begin{aligned} \delta\bar{a} &= -0.43937\,\text{km}, \quad \delta\bar{\theta}_0 = -0.15601\times10^{-2}\,\text{rad} \quad \delta\bar{i} = 0.50006\times10^{-2}\,\text{rad} \\ \delta\bar{q}_{10} &= 0.81355\times10^{-4}, \quad \delta\bar{q}_{20} = 0.13376\times10^{-2}, \quad \delta\bar{\Omega}_0 = 0.21037\times10^{-3}\,\text{rad} \end{aligned} \tag{51}$$

Using the quadratic correction, the corrected mean differential orbital elements, denoted by $\boldsymbol{\delta}\overline{\text{œ}}_2$, are:

$$\begin{aligned} \delta\bar{a} &= 0.19993\,\text{km}, \quad \delta\bar{\theta}_0 = -0.15526\times10^{-2}\,\text{rad} \quad \delta\bar{i} = 0.50000\times10^{-2}\,\text{rad} \\ \delta\bar{q}_{10} &= 0.11479\times10^{-3}, \quad \delta\bar{q}_{20} = 0.13415\times10^{-2}, \quad \delta\bar{\Omega}_0 = 0.19994\times10^{-3}\,\text{rad} \end{aligned} \tag{52}$$

The position and velocity for 10 orbits are obtained by using (51) and the linear part of (47), and (52) with the complete linear-quadratic map in (47). The position error from either map is calculated from the norm of the individual radial, along-track, and out-of-plane errors, and is shown in Figure 4. It is immediately obvious that the linear map quickly loses validity, as indicated by the solid line, since it is observed that the position error is of the same magnitude as the relative orbit size. The error using the linear-quadratic map (broken line) is approximately 10 m after 10 orbits, or approximately 0.03 per cent.

5.3 State Transition with Exact Initial Differential Orbital Elements

In the example considered, the exact mean differential orbital elements are:

$$\begin{aligned} \delta\bar{a} &= 0.20000\,\text{km}, \quad \delta\bar{\theta}_0 = -0.15526\times 10^{-2}\,\text{rad} \quad \delta\bar{i} = 0.50000\times 10^{-2}\,\text{rad} \\ \delta\bar{q}_{10} &= 0.11452\times 10^{-3}, \quad \delta\bar{q}_{20} = 0.13415\times 10^{-2}, \quad \delta\bar{\Omega}_0 = 0.20000\times 10^{-3}\,\text{rad} \end{aligned} \tag{53}$$

Comparison with the $\boldsymbol{\delta\overline{œ}}_1$ and $\boldsymbol{\delta\overline{œ}}_2$, from (51) and (52), respectively shows clearly the advantage of using series reversion and the second-order map to obtain the initial differential orbital elements. Furthermore, the lack of accuracy of $\boldsymbol{\delta\overline{œ}}_1$ results in the large errors for the linear map seen in Figure 4.

If it is now assumed that the initial mean differential orbital elements are known exactly, as given by (53), then propagation of these elements to obtain relative position and velocity results in the errors shown in Figure 5. The solid line, indicating the use of the linear map, shows a marked improvement from Figure 4. The error after 10 orbits is approximately 200 m, which is approximately 0.44 per cent of the orbit size. The error from the second-order map (broken line) does not show improvement over the corresponding indicator in Figure 4, although it is still one order of magnitude lower than that from the linear map. The position error with the inclusion of the quadratic map, as shown in Figure 4 and Figure 5, are nearly identical, because the initial mean differential orbital elements used in both cases, as obtained from (52) and (53) respectively, are nearly equal to each other.

6 Conclusions

A second-order tensor suitable for satellite relative motion calculations has been developed using reversion of a vector series relating differential orbital elements, to relative position and velocity. This tensor can be used to obtain state transition representations, accurate to second-order, and it is shown that its use reduces position errors between analytical propagation and integration by upto two orders of magnitude in comparison with the errors from the linear propagation. The tensor is applicable to cases where a two-body model for satellite motion has been assumed, as well as one perturbed by oblateness effects. In the latter case, no modification to the theory is necessary since the effects of second-order nonlinearities can be of the same order as perturbations due to oblateness effects, and higher-order coupling between these two effects can be neglected. Furthermore, the use of nonsingular elements extends the validity of state transition to any value of eccentricity of an elliptic reference orbit.

Appendix A

Let $g_{1\ldots9}(\theta)$ denote the following functions:

$$
\begin{aligned}
g_1(\theta) &= \frac{\alpha^2}{\eta^3} \\
g_2(\theta) &= \frac{q_2\alpha^2}{(1+\eta)\eta^3} + \frac{\sin\theta\,\alpha}{\eta^2} + \frac{q_2+\sin\theta}{\eta^2} \\
g_3(\theta) &= -\frac{q_1\alpha^2}{(1+\eta)\eta^3} - \frac{\cos\theta\,\alpha}{\eta^2} - \frac{q_1+\cos\theta}{\eta^2} \\
g_4(\theta) &= -\frac{\alpha^3\beta}{\eta^6} \\
g_5(\theta) &= \frac{q_1q_2(3+4\eta)\alpha^2}{2\eta^5(1+\eta)^2} + \frac{\alpha}{\eta^4(1+\eta)}\left[(1+\eta)q_1\sin\theta + \eta q_2\cos\theta\right] + \frac{q_1}{\eta^4}(q_2+\sin\theta) \\
&\quad + \frac{g_2(\theta)}{2\eta^2}\left[(\alpha+1)\cos\theta - \beta\sin\theta - \frac{2q_2\alpha\beta}{\eta(1+\eta)}\right] + \frac{1}{2\eta^2}\sin\theta\,\cos\theta \\
g_6(\theta) &= -\frac{q_1q_2(3+4\eta)\alpha^2}{2\eta^5(1+\eta)^2} - \frac{\alpha}{\eta^4(1+\eta)}\left[\eta q_1\sin\theta + (1+\eta)q_2\cos\theta\right] - \frac{q_2}{\eta^4}(q_1+\cos\theta) \\
&\quad + \frac{g_3(\theta)}{2\eta^2}\left[(\alpha+1)\sin\theta + \beta\cos\theta + \frac{2q_1\alpha\beta}{\eta(1+\eta)}\right] - \frac{1}{2\eta^2}\sin\theta\,\cos\theta \\
g_7(\theta) &= \frac{2\alpha^3}{(1+\eta)\eta^6}\left[q_1 + (1+\eta)\cos\theta\right] - \frac{q_1\alpha^2}{(1+\eta)\eta^6}\left[2\alpha^2 + \eta(1+\eta)\right] \\
g_8(\theta) &= \frac{2\alpha^3}{(1+\eta)\eta^6}\left[q_2 + (1+\eta)\sin\theta\right] - \frac{q_2\alpha^2}{(1+\eta)\eta^6}\left[2\alpha^2 + \eta(1+\eta)\right] \\
g_9(\theta) &= \frac{(q_2^2-q_1^2)}{2\eta^5(1+\eta)^2}\left[(3+4\eta)\alpha^2 + 2\eta(1+\eta)^2\right] - \frac{(q_1\cos\theta - q_2\sin\theta)}{\eta^4(1+\eta)}\left[(1+2\eta)\alpha + 1 + \eta\right] \\
&\quad - \frac{1}{2\eta^2}\cos 2\theta + \frac{(\alpha+1)}{2\eta^2}\left[g_2(\theta)\,\sin\theta + g_3(\theta)\,\cos\theta\right] + \frac{\beta}{2\eta^2}\left[g_2(\theta)\,\cos\theta - g_3(\theta)\,\sin\theta\right] \\
&\quad + \frac{\alpha\beta}{\eta^3(1+\eta)}\left[q_1\,g_2(\theta) - q_2\,g_3(\theta)\right]
\end{aligned}
$$

Using these functions, the non-zero components of **G** are:

$$
\begin{aligned}
G_{11} &= G_{33} = G_{44} = G_{55} = G_{66} = 1 \\
G_{21} &= -\frac{3}{2}\, g_1(\theta_2)\, K(\theta_2, \theta_1) \\
G_{22} &= \frac{g_1(\theta_2)}{g_1\,(\theta_1)} \\
G_{24} &= -\frac{g_1(\theta_2)}{g_1(\theta_1)}\, g_2(\theta_1) + g_2(\theta_2) \\
G_{25} &= -\frac{g_1(\theta_2)}{g_1(\theta_1)}\, g_3(\theta_1) + g_3(\theta_2)
\end{aligned}
$$

The non-zero components of **H** are (with $H_{ijk} = H_{ikj}$):

$$
\begin{aligned}
H_{211} &= \frac{9}{2}\, g_4(\theta_2)\, K^2(\theta_2, \theta_1) + \frac{15}{4}\, g_1(\theta_2)\, K(\theta_2, \theta_1) \\
H_{212} &= -3\, \frac{g_4(\theta_2)}{g_1(\theta_1)}\, K(\theta_2, \theta_1) \\
H_{214} &= -\frac{3}{2}\, g_7\,(\theta_2)\, K(\theta_2, \theta_1) + 3\, \frac{g_4(\theta_2)\, g_2(\theta_1)}{g_1(\theta_1)}\, K(\theta_2, \theta_1) \\
H_{215} &= -\frac{3}{2}\, g_8(\theta_2)\, K(\theta_2, \theta_1) + 3\, \frac{g_4(\theta_2)\, g_3(\theta_1)}{g_1(\theta_1)}\, K(\theta_2, \theta_1) \\
H_{222} &= 2\, \frac{g_4(\theta_2)}{g_1^2(\theta_1)} - 2\, \frac{g_1(\theta_2)\, g_4(\theta_1)}{g_1^3(\theta_1)} \\
H_{224} &= \frac{g_7(\theta_2)}{g_1(\theta_1)} - \frac{g_7(\theta_1)\, g_1(\theta_2) + 2\, g_4(\theta_2)\, g_2(\theta_1)}{g_1^2(\theta_1)} + 2\, \frac{g_1(\theta_2)\, g_4(\theta_1)\, g_2(\theta_1)}{g_1^3(\theta_1)} \\
H_{225} &= \frac{g_8(\theta_2)}{g_1(\theta_1)} - \frac{g_1(\theta_2)\, g_8(\theta_1) + 2\, g_4(\theta_2)\, g_3(\theta_1)}{g_1^2(\theta_1)} + 2\, \frac{g_1(\theta_2)\, g_4(\theta_1)\, g_3(\theta_1)}{g_1^3(\theta_1)} \\
H_{244} &= 2\, g_5(\theta_2) - 2\, \frac{g_5(\theta_1)\, g_1(\theta_2) + g_7(\theta_2)\, g_2(\theta_1)}{g_1(\theta_1)} \\
&\quad + 2\, \frac{g_2(\theta_1)\, \left[g_7(\theta_1)\, g_1(\theta_2) + g_4(\theta_2)\, g_2(\theta_1)\right]}{g_1^2(\theta_1)} - 2\, \frac{g_1(\theta_2)\, g_4(\theta_1)\, g_2^2(\theta_1)}{g_1^3(\theta_1)} \\
H_{245} &= g_9(\theta_2) - \frac{g_1(\theta_2)\, g_9(\theta_1) + g_7(\theta_2)\, g_3(\theta_1) + g_8(\theta_2)\, g_2(\theta_1)}{g_1(\theta_1)} \\
&\quad + \frac{2\, g_4(\theta_2)\, g_3(\theta_1)\, g_2(\theta_1) + g_1(\theta_2)\, g_8(\theta_1)\, g_2(\theta_1) + g_1(\theta_2)\, g_3(\theta_1)\, g_7(\theta_1)}{g_1^2(\theta_1)} \\
&\quad - 2\, \frac{g_1(\theta_2)\, g_4(\theta_1)\, g_3(\theta_1)\, g_2(\theta_1)}{g_1^3(\theta_1)} \\
H_{255} &= 2\, g_6(\theta_2) - 2\, \frac{g_8(\theta_2)\, g_3(\theta_1) + g_1(\theta_2)\, g_6(\theta_1)}{g_1(\theta_1)} \\
&\quad + 2\, \frac{g_3(\theta_1)\, [g_4(\theta_2) g_3(\theta_1) + g_1(\theta_2)\, g_8(\theta_1)]}{g_1^2(\theta_1)} - 2\, \frac{g_1(\theta_2)\, g_4(\theta_1)\, g_3^2(\theta_1)}{g_1^3(\theta_1)}
\end{aligned}
$$

Appendix B

Let $\varepsilon = e\exp(\jmath\omega) = q_1 + \jmath q_2$, and $\tau = \exp(\jmath\theta) = \cos\theta + \jmath\sin\theta$, where $\jmath = \sqrt{-1}$. Furthermore, let ε^* and τ^* denote the complex conjugates of ε and τ, respectively. The non-zero components of the matrices, $\mathbf{P}$, $\mathbf{R}$, and tensors $\mathbf{Q}$ and $\overline{\mathbf{S}}$ (with $Q_{ijk} = Q_{ikj}$ and $\overline{S}_{ijk} = \overline{S}_{ikj}$, respectively), are listed below, by using ε and τ, for conciseness:

$$P_{11} = 1, \quad P_{12} = \frac{\beta}{\alpha}, \quad P_{14} + \jmath P_{15} = -\frac{\tau}{\alpha} - \frac{2\varepsilon}{\eta^2}$$

$$P_{22} = 1, \quad P_{26} = \cos i, \quad P_{33} + \jmath\left(\frac{P_{36}}{\sin i}\right) = -\jmath\tau$$

$$P_{41} = \frac{-3\beta}{2\alpha}, \quad P_{42} = 2 - \frac{3}{\alpha} + \frac{\eta^2}{\alpha^2}, \quad P_{44} + \jmath P_{45} = \frac{3\beta\,\varepsilon}{\eta^2\alpha} - \jmath(\alpha + \jmath\beta)\frac{\tau}{\alpha^2}$$

$$P_{51} = -\frac{3}{2}, \quad P_{52} = -\frac{2\beta}{\alpha}, \quad P_{54} + \jmath P_{55} = \frac{2\tau}{\alpha} + \frac{3\varepsilon}{\eta^2}$$

$$P_{63} + \jmath\left(\frac{P_{66}}{\sin i}\right) = \tau$$

$$R_{11} = -2 + \frac{6\alpha}{\eta^2}, \quad R_{14} + \jmath R_{15} = \frac{2\alpha}{\eta^2}(\beta + \jmath\alpha)$$

$$R_{22} = 1, \quad R_{23} + \jmath R_{26} = \cot i\,\tau^*, \quad R_{33} + \jmath R_{36} = \jmath\tau^*$$

$$R_{41} + \jmath R_{51} = 3(\alpha - \jmath\beta)\,\tau, \quad R_{42} + \jmath R_{52} = \jmath(\alpha - 1 - \jmath\beta)\,\tau$$

$$R_{43} + \jmath R_{53} = \jmath\,\cot i\,\cos\theta(\alpha - 1 - \jmath\beta), \quad R_{44} + \jmath R_{54} = -\jmath\,\alpha\,\tau, \quad R_{45} + \jmath R_{55} = (2\alpha - \jmath\beta)\,\tau$$

$$R_{46} + \jmath R_{56} = -\jmath\,\cot i\,\sin\theta(\alpha - 1 - \jmath\beta), \quad R_{63} + \jmath R_{66} = -\csc i\,\tau^*$$

$$Q_{112} = \frac{\beta}{\alpha}, \quad Q_{114} + \jmath Q_{115} = -\frac{\tau}{\alpha} - \frac{2\varepsilon}{\eta^2}$$

$$Q_{122} = -2 + \frac{3}{\alpha} - \frac{2\eta^2}{\alpha^2}, \quad Q_{124} = +\jmath Q_{125} = -\jmath(\alpha - 2\jmath\beta)\frac{\tau}{\alpha^2} - \frac{2\beta\varepsilon}{\alpha\eta^2}, \quad Q_{126} = -\cos i$$

$$Q_{133} + \jmath\left(\frac{Q_{136}}{\sin i}\right) = \jmath\,\sin\theta\,\tau$$

$$Q_{144} = -\frac{2}{\eta^2} + \frac{4q_1\cos\theta}{\eta^2\alpha} + \frac{2\cos^2\theta}{\alpha^2} \quad Q_{145} = \frac{2(q_1\sin\theta + q_2\cos\theta)}{\eta^2\alpha} + \frac{2\sin\theta\,\cos\theta}{\alpha^2}$$

$$Q_{155} = -\frac{2}{\eta^2} + \frac{4q_2\sin\theta}{\eta^2\alpha} + \frac{2\sin^2\theta}{\alpha^2}$$

$$Q_{212} = 1, \quad Q_{216} = \cos i$$

$$Q_{222} = \frac{2\beta}{\alpha}, \quad Q_{224} + \jmath\, Q_{225} = -\frac{\tau}{\alpha} - \frac{2\varepsilon}{\eta^2}, \quad Q_{226} = \cos i \frac{\beta}{\alpha}$$
$$Q_{233} + \jmath\left(\frac{Q_{236}}{\sin i}\right) = -\sin\theta\,\tau, \quad Q_{246} + \jmath\, Q_{256} = -\cos i\left(\frac{\tau}{\alpha} + \frac{2\varepsilon}{\eta^2}\right)$$

$$Q_{313} + \jmath\left(\frac{Q_{316}}{\sin i}\right) = -\jmath\,\tau, \quad Q_{323} + \jmath\left(\frac{Q_{326}}{\sin i}\right) = (\alpha - \jmath\,\beta)\frac{\tau}{\alpha}$$
$$Q_{334} + \jmath\, Q_{335} = -\sin\theta\left(\frac{\tau}{\alpha} + \frac{2\varepsilon}{\eta^2}\right), \quad Q_{346} + \jmath\, Q_{356} = \sin i\,\cos\theta\left(\frac{\tau}{\alpha} + \frac{2\varepsilon}{\eta^2}\right)$$

$$Q_{411} = \frac{3}{4}\frac{\beta}{\alpha}, \quad Q_{412} = 2 - \frac{3}{2\alpha} + \frac{\eta^2}{\alpha^2}, \quad Q_{414} + \jmath\, Q_{415} = \frac{3\beta\varepsilon}{2\eta^2\alpha} + \jmath\,(\alpha - 2\jmath\,\beta)\frac{\tau}{2\alpha^2}$$
$$Q_{416} = \frac{3}{2}\cos i$$
$$Q_{422} = \frac{\beta}{\alpha}\left(4 - \frac{3}{\alpha} + \frac{2\eta^2}{\alpha^2}\right), \quad Q_{426} = \cos i \frac{2\beta}{\alpha}$$
$$Q_{424} + \jmath\, Q_{425} = -(3\alpha^2 - 4\alpha + 2\eta^2 - \jmath\,\alpha\beta)\frac{\tau}{\alpha^3} - (4\alpha^2 - 3\alpha + 2\eta^2)\frac{\varepsilon}{\eta^2\alpha^2}$$
$$Q_{433} + \jmath\left(\frac{Q_{436}}{\sin i}\right) = \jmath\,\tau^2$$
$$Q_{444} = -\frac{2\beta\,\cos^2\theta}{\alpha^3} - \frac{4q_1\beta\,\cos\theta}{\eta^2\alpha^2} + \frac{1}{\eta^4\alpha}\left[3(1-q_2^2)\beta + 2\eta^2 q_1\sin\theta\right]$$
$$Q_{455} = -\frac{2\beta\,\sin^2\theta}{\alpha^3} - \frac{4q_2\beta\,\sin\theta}{\eta^2\alpha^2} + \frac{1}{\eta^4\alpha}\left[3(1-q_1^2)\beta - 2\eta^2 q_2\cos\theta\right]$$
$$Q_{445} = -\frac{2\beta\,\sin\theta\,\cos\theta}{\alpha^3} - \frac{2}{\eta^2\alpha^2}(q_1\sin\theta + q_2\cos\theta) + \frac{1}{\eta^4\alpha}\left[3q_1q_2\beta + \eta^2(q_2\sin\theta - q_1\cos\theta)\right]$$
$$Q_{446} + \jmath\, Q_{456} = \cos i\left(\frac{2\tau}{\alpha} + \frac{3\varepsilon}{\eta^2}\right)$$

$$Q_{511} = \frac{3}{4}, \quad Q_{512} = -\frac{2\beta}{\alpha}, \quad Q_{514} + \jmath\, Q_{515} = \frac{\tau}{2\alpha} + \frac{3\varepsilon}{2\eta^2}$$
$$Q_{522} = 4 - \frac{8}{\alpha} + \frac{4\eta^2}{\alpha^2}, \quad Q_{524} + \jmath\, Q_{525} = \jmath\,(\alpha - \jmath\,3\beta)\frac{\tau}{\alpha^2} + \frac{4\beta\,\varepsilon}{\eta^2\alpha}, \quad Q_{526} = \cos i\left(2 - \frac{3}{\alpha} + \frac{\eta^2}{\alpha^2}\right)$$
$$Q_{533} + \jmath\left(\frac{Q_{536}}{\sin i}\right) = -\tau^2$$
$$Q_{544} = \frac{3}{\eta^4}(1 - q_2^2) - \frac{2\cos^2\theta}{\alpha^2} - \frac{2q_1\cos\theta}{\eta^2\alpha}, \quad Q_{555} = \frac{3}{\eta^4}(1 - q_1^2) - \frac{2\sin^2\theta}{\alpha^2} - \frac{2q_2\sin\theta}{\eta^2\alpha}$$
$$Q_{545} = \frac{3q_1q_2}{\eta^4} - \frac{2\sin\theta\,\cos\theta}{\alpha^2} - \frac{1}{\eta^2\alpha}(q_1\sin\theta + q_2\cos\theta)$$
$$Q_{546} + \jmath\, Q_{556} = \cos i \frac{3\beta\,\varepsilon}{\eta^2\alpha} - \jmath\,\cos i\,(\alpha + \jmath\,\beta)\frac{\tau}{\alpha^2}$$

$$
\begin{aligned}
Q_{613} + \jmath \left(\frac{Q_{616}}{\sin i} \right) &= -(\alpha - \jmath\, 3\beta) \frac{\tau}{2\alpha} \\
Q_{623} + \jmath \left(\frac{Q_{626}}{\sin i} \right) &= -\jmath\, (\alpha^2 - 3\alpha + \eta^2 - \jmath\, \alpha\beta) \frac{\tau}{\alpha^2} \\
Q_{634} + \jmath\, Q_{635} &= \frac{1}{\alpha} + (\alpha \cos\theta + 3\beta \sin\theta) \frac{\varepsilon}{\eta^2 \alpha} + \beta \sin\theta \frac{\tau}{\alpha^2} \\
Q_{646} + \jmath\, Q_{656} &= \jmath\, \frac{\sin i}{\alpha} + \sin i\, (\alpha \sin\theta - 3\beta \cos\theta) \frac{\varepsilon}{\eta^2 \alpha} - \sin i\, \beta \cos\theta \frac{\tau}{\alpha^2}
\end{aligned}
$$

$$
Q_{166} + \jmath\, Q_{266} = \jmath\, \sin^2 i\, \sin\theta\, \tau^* - 1, \quad Q_{466} + \jmath\, Q_{566} = -\jmath\, \sin^2 i\, \tau^2, \quad Q_{366} + \jmath\, Q_{666} = \jmath\, \sin i\, \cos i\, \tau^*
$$

$$
\begin{aligned}
\overline{S}_{111} &= \frac{6}{\eta^4} (\eta^2 - 2\alpha)(\eta^2 - 6\alpha), \quad \overline{S}_{114} = -\frac{6\alpha\beta}{\eta^4} (\eta^2 - 4\alpha), \quad \overline{S}_{115} = -\frac{6\alpha^2}{\eta^4} (\eta^2 - 4\alpha) \\
\overline{S}_{122} &= -\frac{2}{\eta^2} (\eta^2 - 3\alpha), \quad \overline{S}_{124} = -\frac{2\alpha^2}{\eta^2}, \quad \overline{S}_{125} = \frac{2\alpha\beta}{\eta^2} \\
\overline{S}_{133} &= -\frac{2}{\eta^2} (\eta^2 - 3\alpha + \alpha^2), \quad \overline{S}_{136} = \frac{2\alpha\beta}{\eta^2} \\
\overline{S}_{144} &= -\frac{2\alpha^2}{\eta^2} (3\eta^2 - 8\alpha + 4\alpha^2), \quad \overline{S}_{145} = \frac{8\alpha^3\beta}{\eta^4} \\
\overline{S}_{155} &= \frac{2\alpha^2}{\eta^4} (\eta^2 + 4\alpha^2) \\
\overline{S}_{166} &= \frac{2\alpha^2}{\eta^2}
\end{aligned}
$$

$$
\overline{S}_{212} = -1, \quad \overline{S}_{213} + \jmath\, \overline{S}_{216} = -\cot i\, \tau^*, \quad \overline{S}_{223} + \jmath\, \overline{S}_{226} = -\jmath\, \cot i\, \tau^*
$$

$$
\overline{S}_{233} = -(\cot^2 i + \csc^2 i) \sin\theta \cos\theta, \quad \overline{S}_{234} = \cot i \sin\theta, \quad \overline{S}_{236} = -\cot^2 i \cos^2\theta + \csc^2 i \sin^2\theta
$$

$$
\overline{S}_{256} = \cot i \sin\theta
$$

$$
\overline{S}_{266} = (\cot^2 i + \csc^2 i) \sin\theta \cos\theta
$$

$$
\overline{S}_{313} + \jmath\, \overline{S}_{316} = -\jmath\, \tau^*, \quad \overline{S}_{323} + \jmath \overline{S}_{326} = \tau^*, \quad \overline{S}_{333} + \jmath\, \overline{S}_{336} = \cot i \cos\theta\, \tau^*
$$

$$
\overline{S}_{356} = -\cos\theta, \quad \overline{S}_{366} = \cot i \sin^2\theta
$$

$$\overline{S}_{411} + \jmath\,\overline{S}_{511} = 6(\tau+\varepsilon), \quad \overline{S}_{412} + \jmath\,\overline{S}_{512} = \jmath\,(3\tau+2\varepsilon), \quad \overline{S}_{413} + \jmath\,\overline{S}_{513} = \jmath\,\cot i\,\cos\theta\,(3\tau+2\varepsilon)$$

$$\overline{S}_{414} + \jmath\,\overline{S}_{514} = -2\jmath\,\alpha\,\tau, \quad \overline{S}_{415} + \jmath\,\overline{S}_{515} = 2(\alpha+1)\,\tau - 2\varepsilon, \quad \overline{S}_{416} + \jmath\,\overline{S}_{516} = -\jmath\,\cot i\,\sin\theta\,(3\tau+2\varepsilon)$$

$$\overline{S}_{422} + \jmath\,\overline{S}_{522} = (3\tau+2\varepsilon), \quad \overline{S}_{423} + \jmath\,\overline{S}_{523} = -\cot i\,\tau\,\varepsilon, \quad \overline{S}_{424} + \jmath\,\overline{S}_{524} = -(\tau+\varepsilon)$$

$$\overline{S}_{433} + \jmath\,\overline{S}_{533} = (\tau+2\varepsilon) - \frac{1}{2}\csc^2 i\,\varepsilon\,(1+\tau^2) + \frac{1}{2}(\varepsilon-\varepsilon^*)\tau^2 - \frac{\csc^2 i}{4}\,\varepsilon\,(\tau^2-\tau^{*2})$$

$$\overline{S}_{434} + \jmath\,\overline{S}_{534} = \cot i\,\left(\alpha\,\cos\theta\,\tau + \jmath\,\sin\theta\,\varepsilon\right), \quad \overline{S}_{435} + \jmath\,\overline{S}_{535} = \jmath\,\cot i\,\cos\theta\,[(\alpha+1)\tau+\varepsilon]$$

$$\overline{S}_{436} + \jmath\,\overline{S}_{536} = -\jmath\,\tau - \jmath\,\frac{1}{2}\csc^2 i\,\varepsilon\,\tau^2 + \frac{1}{2}\jmath\,(\varepsilon-\varepsilon^*)\tau^2 + \jmath\,\frac{\csc^2 i}{4}\,\varepsilon\,(\tau^2-\tau^{*2})$$

$$\overline{S}_{425} + \jmath\,\overline{S}_{525} = \jmath\,(\tau+\varepsilon), \quad \overline{S}_{426} + \jmath\,\overline{S}_{526} = -\jmath\,\cot i\,\tau\,\varepsilon$$

$$\overline{S}_{445} + \jmath\,\overline{S}_{545} = -\jmath\,\alpha\,\tau, \quad \overline{S}_{446} + \jmath\,\overline{S}_{546} = -\cot i\,\sin\theta\,\alpha\,\tau$$

$$\overline{S}_{455} + \jmath\,\overline{S}_{555} = 2\alpha\,\tau, \quad \overline{S}_{456} + \jmath\,\overline{S}_{556} = -\jmath\,\cot i\,(\alpha+1)\,\sin\theta\,\tau$$

$$\overline{S}_{466} + \jmath\,\overline{S}_{566} = 2(\tau+\varepsilon) - \frac{1}{2}\csc^2 i\,\varepsilon\,(1-\tau^2) - \frac{1}{2}(\varepsilon-\varepsilon^*)\tau^2 + \frac{\csc^2 i}{4}\,\varepsilon\,(\tau^2-\tau^{*2})$$

$$\overline{S}_{613} + \jmath\,\overline{S}_{616} = \csc i\,\tau^*, \quad \overline{S}_{623} + \jmath\,\overline{S}_{626} = \jmath\,\csc i\,\tau^*, \quad \overline{S}_{633} + \jmath\,\overline{S}_{636} = \jmath\,\cot i\,\csc i\,\tau^{*2}$$

$$\overline{S}_{656} = -\csc i\,\sin\theta, \quad \overline{S}_{666} = -\cot i\,\csc i\,\sin 2\theta$$

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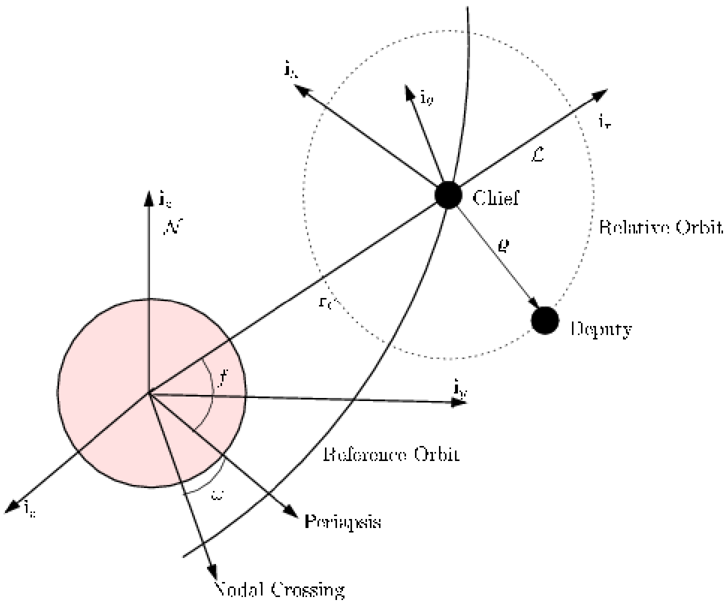


Fig. 1 Frames of Reference

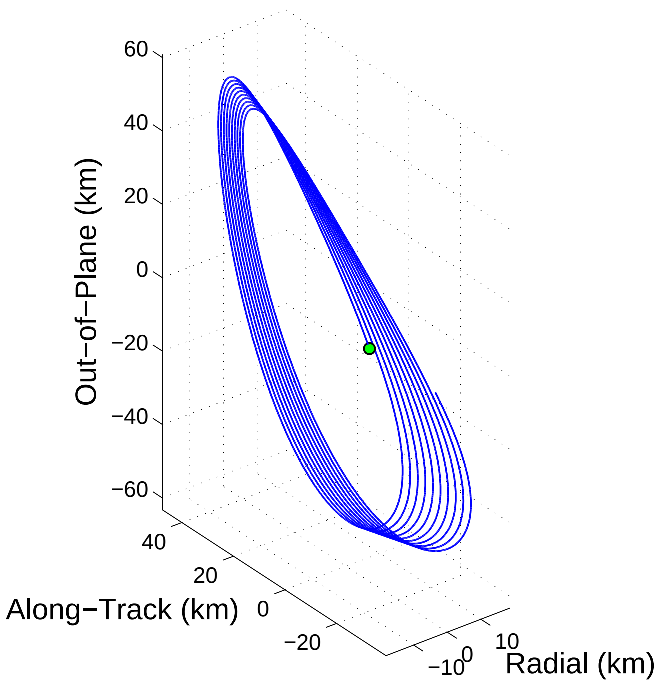
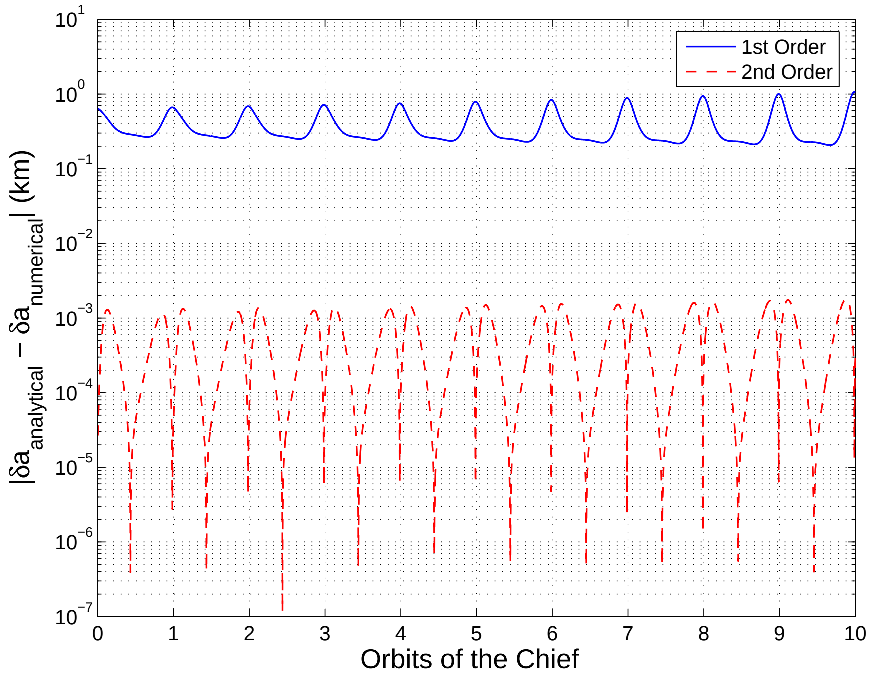
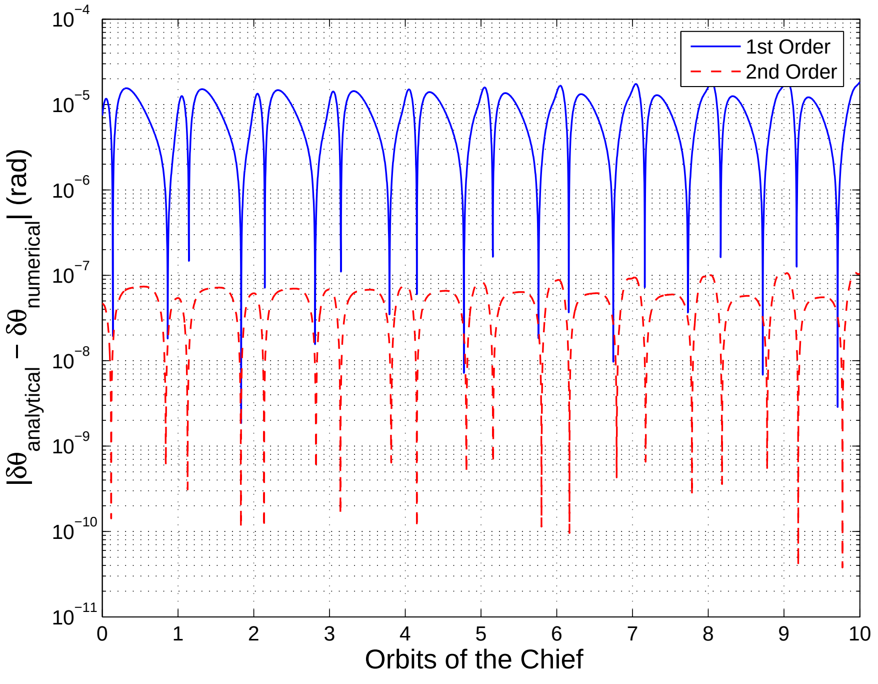


Fig. 2 Trajectory in the Rotating Frame

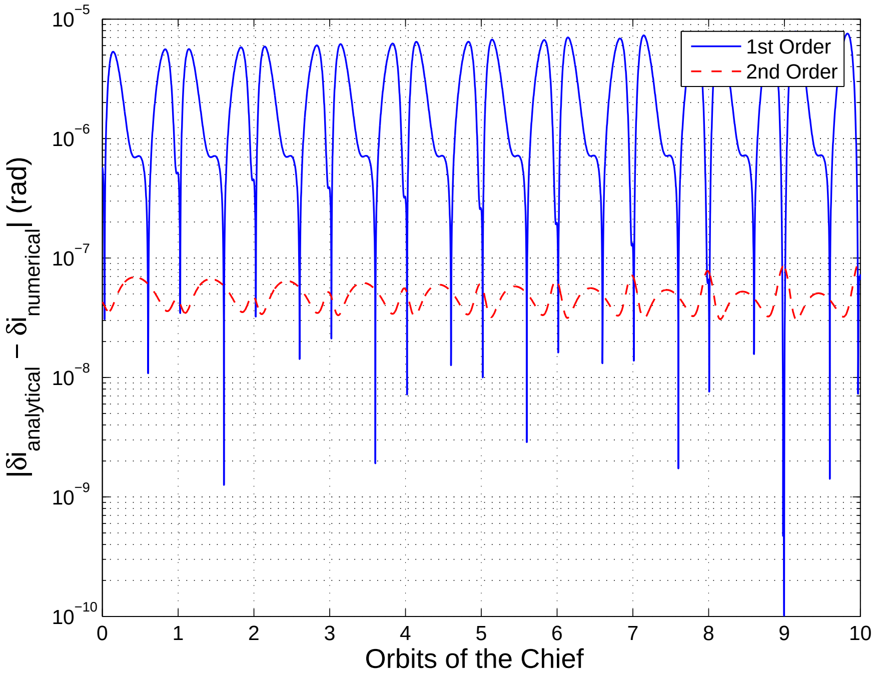


(a) Differential Semimajor Axis Error

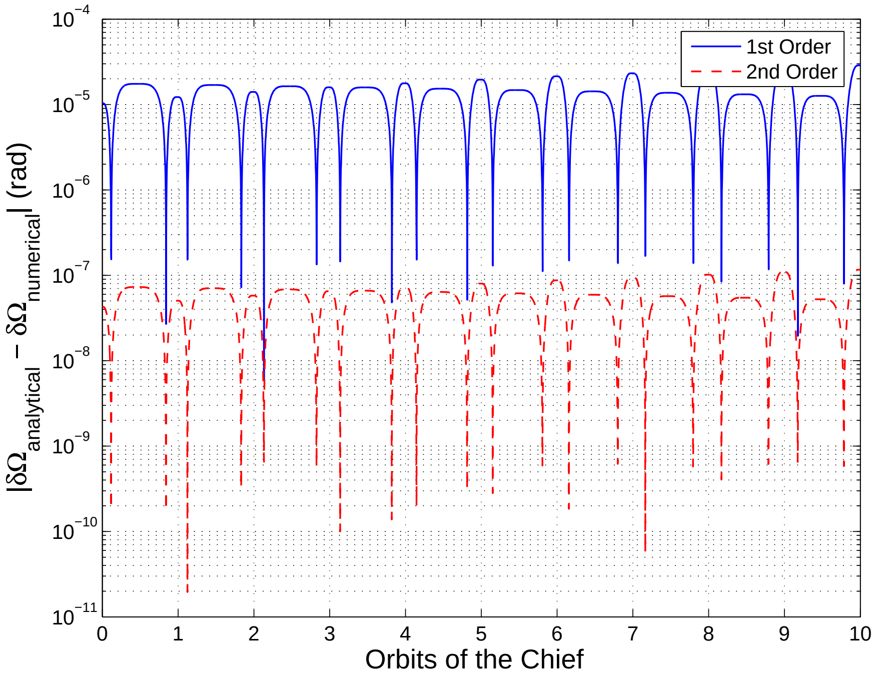


(b) Differential True Argument of Latitude Error

Fig. 3 Errors in Differential Orbital Elements Obtained from First-Order and Second-Order Maps

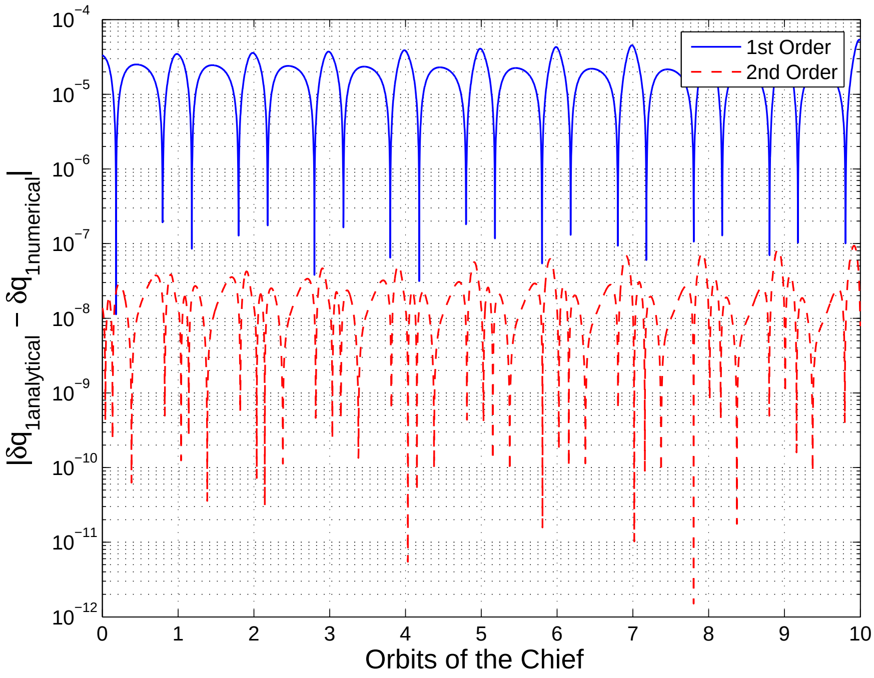


(c) Differential Inclination Error

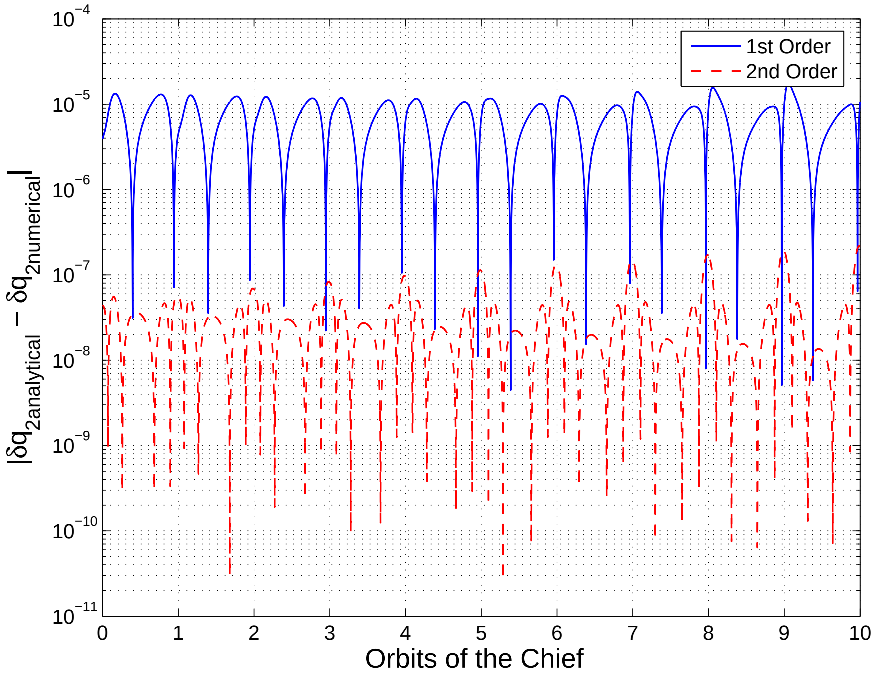


(d) Differential RAAN Error

Fig. 3 Errors in Differential Orbital Elements Obtained from First-Order and Second-Order Maps



(e) δq_1 Error



(f) δq_2 Error

Fig. 3 Errors in Differential Orbital Elements Obtained from First-Order and Second-Order Maps

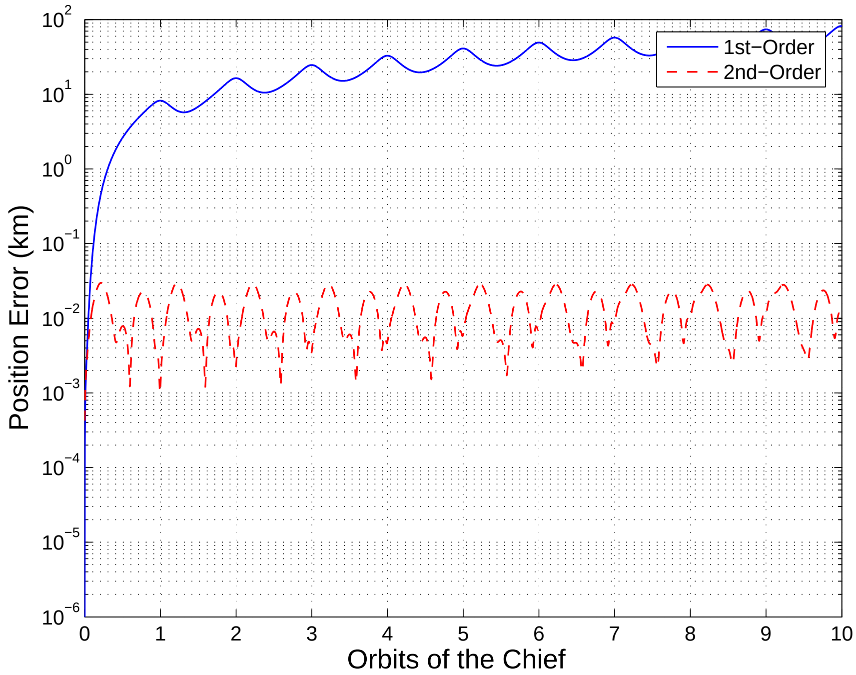


Fig. 4 Position Errors from First-Order and Second-Order STTs, with Oblateness Effects

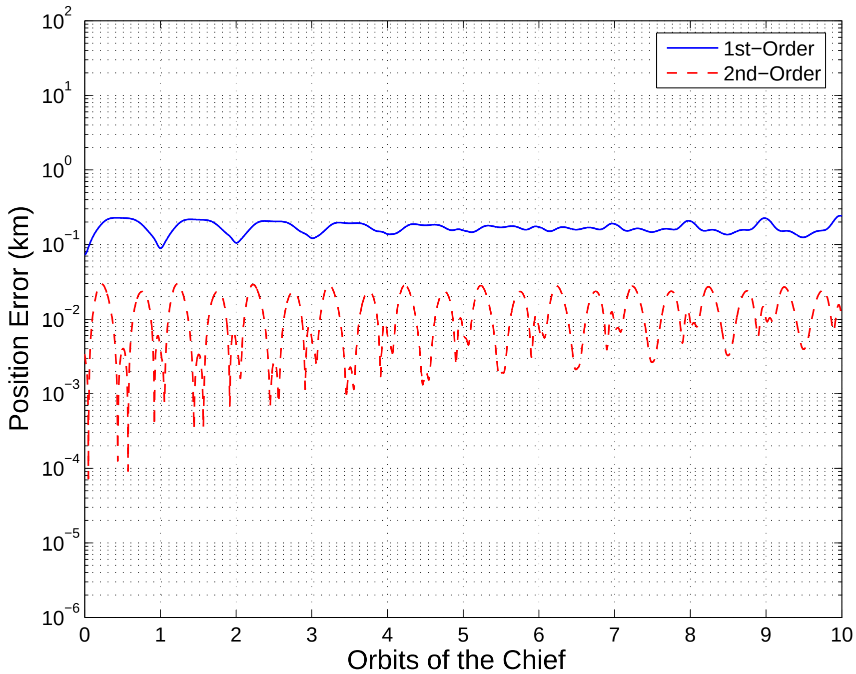


Fig. 5 Position Errors from First-Order and Second-Order STTs, with Oblateness Effects, with Accurate Initial Conditions