Paper AAS 04-163

DEVELOPMENT OF A STATE TRANSITION MATRIX FOR RELATIVE MOTION USING THE UNIT SPHERE APPROACH

Hui Yan Prasenjit Sengupta Srinivas R. Vadali and Kyle T. Alfriend

Department of Aerospace Engineering
Texas A&M University, College Station, TX 77843-3141

14th AAS/AIAA Space Flight Mechanics Conference

Maui, Hawaii, **February 8-12, 2004**

AAS Publications Office, P.O. Box 28130, San Diego, CA 92198

AAS 04-163

DEVELOPMENT OF A STATE TRANSITION MATRIX FOR RELATIVE MOTION USING THE UNIT SPHERE APPROACH

Hui Yan[1], Prasenjit Sengupta[2], Srinivas R. Vadali[3] and Kyle T. Alfriend[4]

With the desire to be able to have accurate solutions for relative motions, Hill's equations are insufficient for long-term prediction due to the spherical Earth, circular reference orbit and linearization assumptions. Using differential orbital elements, the Gim-Alfriend state transition matrix (STM) incorporates 1st order J_2 and eccentricity effects in the equations with the assumption of linearization. In this paper, we use the unit sphere method to get the exact kinematic description for relative positions and develop a high accuracy STM for relative motions.

INTRODUCTION

The analysis of the relative motion of satellites began with the paper by Clohessy and Wiltshire [1] in 1960, who derived the equations of motion for one satellite relative to another when the reference satellite is in a circular orbit about a spherical Earth. They also assumed that the separation distance between the satellites is small compared to the orbit radius so that the equations of motion could be linearized. These are sometimes called Hill's equations because Hill used the same approach in his research on the motion [2] of the moon. Lawden [3] and Tschauner and Hempel [4] obtained independently, the solution to the linearized equations of motion when the reference satellite is in an elliptical orbit. After that, Cater [5] and Melton [6] developed more efficient solutions of the relative motion. The characteristics of the above approach is to linearize the nonlinear equations then propagate the states in the Cartesian frame to obtain the STM. In 1995, Garrison et al [7] used a novel method to propagate the states in the orbital elements space then transform the differential orbital elements into the Cartesian coordinates with the linearization. The linear approximation using the differential orbital elements is more accurate than that using the Cartesian or curvilinear coordinates as shown in Ref [8]. Recently the research interest has been focused on formation flying so that the long-time accuracy of the bounded solutions of relative motion is a major concern. Although several solutions exist for the unperturbed non-circular reference orbit problem, state

---

[1]Graduate Student, Department of Aerospace Engineering, 3141 TAMU, Texas A&M University, College Station, TX 77843-3141, Email: hyan@tamu.edu
[2]Graduate Student, Department of Aerospace Engineering, 3141 TAMU, Texas A&M University, College Station, TX 77843-3141, Email: prasenjit@tamu.edu
[3]Stewart & Stevenson Professor, Department of Aerospace Engineering, 3141 TAMU, Texas A&M University, College Station, TX 77843-3141, Email: svadali@aero.tamu.edu,
[4]Wisenbaker II Chair Professor, Department of Aerospace Engineering, 3141 TAMU, Texas A&M University, College Station, TX 77843-3141, Email: alfriend@aero.tamu.edu

transition matrices including perturbations due to J_2 and geometric nonlinearity effects still remain elusive. Using differential orbital elements, Alfriend et al. [9] described relative motion in terms of mean elements, incorporating 1st order J_2 and eccentricity effects in the equations. Although the mean orbital elements can be analytically propagated in the mean space, we need to transfer the mean orbital elements into positions and velocities in the LVLH Cartesian frames. Gim and Alfriend [10] used a geometric method and Brouwer theory [11] to complete the transformation and get the STM of the relative motion for the perturbed non-circular reference orbit. The Gim and Alfriend's STM was obtained by considering the relative motion as a result of small changes in the orbital elements of the Deputy with respect to those of the Chief.

In this paper, we use the unit sphere approach, proposed by Vadali [12], to establish a state transition matrix for the perturbed non-circular reference orbit problem. In the unit sphere approach, the relative motion problem is studied by projecting the motion of the two satellites onto a unit sphere. This is achieved by normalizing the position vector of each satellite with respect to its radius. This process allows one to study the relative motion using spherical trigonometry so that a kinematically exact description is obtained for the relative positions in terms of the differential orbital elements, without recourse to linearization. In order to obtain time-explicit expressions, the method requires the solution of Kepler's equation or eccentricity expansions to obtain the radial distance and argument of latitude. Taking time derivatives for the relative positions, we get analytical expressions for the relative velocities with the help of Gauss' equations. However, we do not find the linearly inverse analytical expressions for the relative motion. This is why we develop the linear STM based on the unit sphere approach. Our numerical evaluations using a nonlinearity index [13] show that this approach has a very high accuracy, as compared with Gim-Alfriend solutions.

UNIT SPHERE APPROACH

The relative position on the unit sphere is given by

$$\begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix} = \left[C_C C_D^T - I \right] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{1}$$

where Δx, Δy, and Δz, are respectively, the radial, along-track, and cross-track relative positions on the unit sphere, C_c and C_D are the direction cosine matrices of the Chief and Deputy, and the subscripts C and D represent the Chief and Deputy, respectively. This results in analytical expressions for the so-called "sub-satellite" points that are functions of the angles only (right ascension Ω, inclination i, and argument of latitude θ). Equation (1) can be expanded as

$$\begin{aligned}\Delta x = -1 + c^2\left(0.5 i_C\right)c^2\left(0.5 i_D\right)c\left(\theta_D - \theta_C + \Omega_D - \Omega_C\right) \\ + s^2\left(0.5 i_C\right)s^2\left(0.5 i_D\right)c\left(\theta_D - \theta_C - \Omega_D + \Omega_C\right) \\ + s^2\left(0.5 i_C\right)c^2\left(0.5 i_D\right)c\left(\theta_D + \theta_C + \Omega_D - \Omega_C\right) \\ + c^2\left(0.5 i_C\right)s^2\left(0.5 i_D\right)c\left(\theta_D + \theta_C - \Omega_D + \Omega_C\right) \\ + 0.5 s\left(i_C\right)s\left(i_D\right)\left[c\left(\theta_D - \theta_C\right) - c\left(\theta_D + \theta_C\right)\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\Delta y = c^2\left(0.5 i_C\right)c^2\left(0.5 i_D\right)s\left(\theta_D - \theta_C + \Omega_D - \Omega_C\right) \\ + s^2\left(0.5 i_C\right)s^2\left(0.5 i_D\right)s\left(\theta_D - \theta_C - \Omega_D + \Omega_C\right) \\ - s^2\left(0.5 i_C\right)c^2\left(0.5 i_D\right)s\left(\theta_D + \theta_C + \Omega_D - \Omega_C\right) \\ - c^2\left(0.5 i_C\right)s^2\left(0.5 i_D\right)s\left(\theta_D + \theta_C - \Omega_D + \Omega_C\right) \\ + 0.5 s\left(i_C\right)s\left(i_D\right)\left[s\left(\theta_D - \theta_C\right) - s\left(\theta_D + \theta_C\right)\right]\end{aligned} \tag{3}$$

$$\Delta z = -s\left(i_C\right)s\left(\Omega_D - \Omega_C\right)c\left(\theta_D\right) - \left[s\left(i_C\right)c\left(i_D\right)c\left(\Omega_D - \Omega_C\right) - c\left(i_C\right)s\left(i_D\right)\right]s\left(\theta_D\right) \tag{4}$$

The actual relative positions between the two satellites are

$$\delta x = r_D\left(1 + \Delta x\right) - r_C \tag{5}$$

$$\delta y = r_D \Delta y \tag{6}$$

$$\delta z = r_D \Delta z \tag{7}$$

Taking time derivatives, we have

$$\delta \dot{x} = \dot{r}_D\left(1 + \Delta x\right) + r_D \Delta \dot{x} - \dot{r}_C \tag{8}$$

$$\delta \dot{y} = \dot{r}_D \Delta y + r_D \Delta \dot{y} \tag{9}$$

$$\delta \dot{z} = \dot{r}_D \Delta z + r_D \Delta \dot{z} \tag{10}$$

GIM-ALFRIEND STATE TRANSITION MATRIX

To avoid the singularity due to the Gauss equation when eccentricity is zero, choose the nonsingular orbital elements $\boldsymbol{e} = \left(a, \theta, i, q_1, q_2, \Omega\right)$ and

$$q_1 = e\cos\omega$$

$$q_2 = e\sin\omega$$

where a is the semi-major axis and ω is the argument of perigee. Assume $\bar{\boldsymbol{e}}$ represents the nonsingular element vectors in the mean space. Let $\bar{\phi}_{\bar{e}}(t,t_0)$ be the STM for the relative mean variables and let $D(t)$ be the transformation matrix from the relative mean variables to the relative osculating variables

$$\delta\bar{\boldsymbol{e}}(t) = \bar{\phi}_{\bar{e}}(t,t_0)\delta\bar{\boldsymbol{e}}(t_0) \tag{11}$$

$$\delta\boldsymbol{e}(t) = D(t)\delta\bar{\boldsymbol{e}}(t) = D(t)\bar{\phi}_{\bar{e}}(t,t_0)D^{-1}(t_0)\delta\boldsymbol{e}(t_0) \tag{12}$$

$$\delta\boldsymbol{X} = \Sigma(t)\delta\boldsymbol{e}(t) \tag{13}$$

Thus, the relative state at any time can be expressed as follows [6]

$$\delta\boldsymbol{X}(\boldsymbol{t}) = \Phi(t,t_0)\delta\boldsymbol{X}(t_0) \tag{14}$$

$$\Phi(t,t_0) = \Sigma(t)D(t)\bar{\phi}_{\bar{e}}(t,t_0)D^{-1}(t_0)\Sigma^{-1}(t_0) \tag{15}$$

where $\Phi(t,t_0)$ is the STM for the relative motion and $\Sigma(t)$ is called the transformation matrix that is derived by the geometric method.

A NEW STATE TRANSFORMATION MATRIX

In this section, we derive the transformation matrix using the unit sphere approach. Gauss' equations in terms of the nonsingular elements with the perturbing accelerations in the Local-Vertical Local-Horizontal (LVLH) frame are

$$\dot{a} = \frac{2a^2}{h}\left[(q_1\sin\theta - q_2\cos\theta)u_r + \frac{p}{r}u_\theta\right] \tag{16}$$

$$\dot{\theta} = \frac{h}{r^2} - \frac{r\sin\theta\cos i}{h\sin i}u_h \tag{17}$$

$$\dot{i} = \frac{r\cos\theta}{h}u_h \tag{18}$$

$$\dot{q}_1 = \frac{p\sin\theta}{h}u_r + \frac{(p+r)\cos\theta + q_1 r}{h}u_\theta + \frac{q_2 r\sin\theta\cos i}{h\sin i}u_h \tag{19}$$

$$\dot{q}_2 = -\frac{p\cos\theta}{h}u_r + \frac{(p+r)\sin\theta + q_2 r}{h}u_\theta - \frac{q_1 r\sin\theta\cos i}{h\sin i}u_h \tag{20}$$

$$\dot{\Omega} = \frac{r\sin\theta}{h\sin i}u_h \tag{21}$$

where

$$p = a\left(1 - q_1^2 - q_2^2\right) \tag{22}$$

$$h = \sqrt{\mu p} \tag{23}$$

$$r = \frac{p}{1 + q_1 \cos\theta + q_2 \sin\theta} \tag{24}$$

Considering the gravity perturbation J_2, the accelerations are

$$u_r = -1.5\frac{J_2 \mu R_e^2}{r^4}\left(1 - 3\sin^2 i \sin^2 \theta\right) \tag{25}$$

$$u_\theta = -1.5\frac{J_2 \mu R_e^2 \sin^2 i \sin 2\theta}{r^4} \tag{26}$$

$$u_h = -1.5\frac{J_2 \mu R_e^2 \sin 2i \sin\theta}{r^4} \tag{27}$$

From Eqs.(2-4), we have

$$\Delta\dot{x} = \frac{\partial \Delta x}{\partial \theta_C}\dot{\theta}_C + \frac{\partial \Delta x}{\partial \theta_D}\dot{\theta}_D + \frac{\partial \Delta x}{\partial i_C}\dot{i}_C + \frac{\partial \Delta x}{\partial i_D}\dot{i}_D + \frac{\partial \Delta x}{\partial \Omega_C}\dot{\Omega}_C + \frac{\partial \Delta x}{\partial \Omega_D}\dot{\Omega}_D \tag{28}$$

$$\Delta\dot{y} = \frac{\partial \Delta y}{\partial \theta_C}\dot{\theta}_C + \frac{\partial \Delta y}{\partial \theta_D}\dot{\theta}_D + \frac{\partial \Delta y}{\partial i_C}\dot{i}_C + \frac{\partial \Delta y}{\partial i_D}\dot{i}_D + \frac{\partial \Delta y}{\partial \Omega_C}\dot{\Omega}_C + \frac{\partial \Delta y}{\partial \Omega_D}\dot{\Omega}_D \tag{29}$$

$$\Delta\dot{z} = \frac{\partial \Delta z}{\partial \theta_C}\dot{\theta}_C + \frac{\partial \Delta z}{\partial \theta_D}\dot{\theta}_D + \frac{\partial \Delta z}{\partial i_C}\dot{i}_C + \frac{\partial \Delta z}{\partial i_D}\dot{i}_D + \frac{\partial \Delta z}{\partial \Omega_C}\dot{\Omega}_C + \frac{\partial \Delta z}{\partial \Omega_D}\dot{\Omega}_D \tag{30}$$

Since $\boldsymbol{e}_{\mathrm{D}} = \boldsymbol{e}_{\mathrm{C}} + \Delta \boldsymbol{e}$

$$\begin{aligned} \frac{\partial \Delta\dot{x}}{\partial \Delta\mathbf{e}} = & \frac{\partial^2 \Delta x}{\partial \theta_C \partial \mathbf{e}_D}\dot{\theta}_C + \frac{\partial^2 \Delta x}{\partial \theta_D \partial \mathbf{e}_D}\dot{\theta}_D + \frac{\partial \Delta x}{\partial \theta_D}\frac{\partial \dot{\theta}_D}{\partial \mathbf{e}_D} \\ & + \frac{\partial^2 \Delta x}{\partial i_C \partial \mathbf{e}_D}\dot{i}_C + \frac{\partial^2 \Delta x}{\partial i_D \partial \mathbf{e}_D}\dot{i}_D + \frac{\partial \Delta x}{\partial i_D}\frac{\partial \dot{i}_D}{\partial \mathbf{e}_D} \\ & + \frac{\partial^2 \Delta x}{\partial \Omega_C \partial \mathbf{e}_D}\dot{\Omega}_C + \frac{\partial^2 \Delta x}{\partial \Omega_D \partial \mathbf{e}_D}\dot{\Omega}_D + \frac{\partial \Delta x}{\partial \Omega_D}\frac{\partial \dot{\Omega}_D}{\partial \mathbf{e}_D} \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial \Delta\dot{y}}{\partial \Delta\mathbf{e}} = & \frac{\partial^2 \Delta y}{\partial \theta_C \partial \mathbf{e}_D}\dot{\theta}_C + \frac{\partial^2 \Delta y}{\partial \theta_D \partial \mathbf{e}_D}\dot{\theta}_D + \frac{\partial \Delta y}{\partial \theta_D}\frac{\partial \dot{\theta}_D}{\partial \mathbf{e}_D} \\ & + \frac{\partial^2 \Delta y}{\partial i_C \partial \mathbf{e}_D}\dot{i}_C + \frac{\partial^2 \Delta y}{\partial i_D \partial \mathbf{e}_D}\dot{i}_D + \frac{\partial \Delta y}{\partial i_D}\frac{\partial \dot{i}_D}{\partial \mathbf{e}_D} \\ & + \frac{\partial^2 \Delta y}{\partial \Omega_C \partial \mathbf{e}_D}\dot{\Omega}_C + \frac{\partial^2 \Delta y}{\partial \Omega_D \partial \mathbf{e}_D}\dot{\Omega}_D + \frac{\partial \Delta y}{\partial \Omega_D}\frac{\partial \dot{\Omega}_D}{\partial \mathbf{e}_D} \end{aligned} \tag{32}$$

$$\frac{\partial \Delta \dot{z}}{\partial \Delta \mathbf{e}} = \frac{\partial^2 \Delta z}{\partial \theta_C \partial \mathbf{e}_D} \dot{\theta}_C + \frac{\partial^2 \Delta z}{\partial \theta_D \partial \mathbf{e}_D} \dot{\theta}_D + \frac{\partial \Delta z}{\partial \theta_D} \frac{\partial \dot{\theta}_D}{\partial \mathbf{e}_D}$$
$$+ \frac{\partial^2 \Delta z}{\partial i_C \partial \mathbf{e}_D} \dot{i}_C + \frac{\partial^2 \Delta z}{\partial i_D \partial \mathbf{e}_D} \dot{i}_D + \frac{\partial \Delta z}{\partial i_D} \frac{\partial \dot{i}_D}{\partial \mathbf{e}_D} \tag{33}$$
$$+ \frac{\partial^2 \Delta z}{\partial \Omega_C \partial \mathbf{e}_D} \dot{\Omega}_C + \frac{\partial^2 \Delta z}{\partial \Omega_D \partial \mathbf{e}_D} \dot{\Omega}_D + \frac{\partial \Delta z}{\partial \Omega_D} \frac{\partial \dot{\Omega}_D}{\partial \mathbf{e}_D}$$

From Eqs.(22-24), we have

$$\dot{r}_D = \frac{\partial r_D}{\partial a_D} \dot{a}_D + \frac{\partial r_D}{\partial \theta_D} \dot{\theta}_D + \frac{\partial r_D}{\partial q_{1D}} \dot{q}_{1D} + \frac{\partial r_D}{\partial q_{2D}} \dot{q}_{2D} \tag{34}$$

So,

$$\frac{\partial \dot{r}_D}{\partial \Delta \mathbf{e}} = \frac{\partial^2 r_D}{\partial a_D \partial \mathbf{e}_D} \dot{a}_D + \frac{\partial r_D}{\partial a_D} \frac{\partial \dot{a}_D}{\partial \mathbf{e}_D} + \frac{\partial^2 r_D}{\partial \theta_D \partial \mathbf{e}_D} \dot{\theta}_D$$
$$+ \frac{\partial r_D}{\partial \theta_D} \frac{\partial \dot{\theta}_D}{\partial \mathbf{e}_D} + \frac{\partial^2 r_D}{\partial q_{1D} \partial \mathbf{e}_D} \dot{q}_{1D} + \frac{\partial r_D}{\partial q_{1D}} \frac{\partial \dot{q}_{1D}}{\partial \mathbf{e}_D} \tag{35}$$
$$+ \frac{\partial^2 r_D}{\partial q_{2D} \partial \mathbf{e}_D} \dot{q}_{2D} + \frac{\partial r_D}{\partial q_{2D}} \frac{\partial \dot{q}_{2D}}{\partial \mathbf{e}_D}$$

Now we obtain the transformation matrix

$$\Sigma(t) = \begin{pmatrix} \frac{\partial \delta x}{\partial \Delta \mathbf{e}} & \frac{\partial \delta \dot{x}}{\partial \Delta \mathbf{e}} & \frac{\partial \delta y}{\partial \Delta \mathbf{e}} & \frac{\partial \delta \dot{y}}{\partial \Delta \mathbf{e}} & \frac{\partial \delta z}{\partial \Delta \mathbf{e}} & \frac{\partial \delta \dot{z}}{\partial \Delta \mathbf{e}} \end{pmatrix}^T \tag{36}$$

where

$$\frac{\partial \delta x}{\partial \Delta \mathbf{e}} = \frac{\partial r_D}{\partial \Delta \mathbf{e}} (1 + \Delta x) + r_D \frac{\partial \Delta x}{\partial \Delta \mathbf{e}} \tag{37}$$

$$\frac{\partial \delta \dot{x}}{\partial \Delta \mathbf{e}} = \frac{\partial \dot{r}_D}{\partial \Delta \mathbf{e}} (1 + \Delta x) + \dot{r}_D \frac{\partial \Delta x}{\partial \Delta \mathbf{e}} + \frac{\partial r_D}{\partial \Delta \mathbf{e}} \Delta \dot{x} + r_D \frac{\partial \Delta x}{\partial \Delta \mathbf{e}} \tag{38}$$

$$\frac{\partial \delta y}{\partial \Delta \mathbf{e}} = \frac{\partial r_D}{\partial \Delta \mathbf{e}} \Delta y + r_D \frac{\partial \Delta y}{\partial \Delta \mathbf{e}} \tag{37}$$

$$\frac{\partial \delta \dot{y}}{\partial \Delta \mathbf{e}} = \frac{\partial \dot{r}_D}{\partial \Delta \mathbf{e}} \Delta y + \dot{r}_D \frac{\partial \Delta y}{\partial \Delta \mathbf{e}} + \frac{\partial r_D}{\partial \Delta \mathbf{e}} \Delta \dot{y} + r_D \frac{\partial \Delta y}{\partial \Delta \mathbf{e}} \tag{38}$$

$$\frac{\partial \delta z}{\partial \Delta \mathbf{e}} = \frac{\partial r_D}{\partial \Delta \mathbf{e}} \Delta z + r_D \frac{\partial \Delta z}{\partial \Delta \mathbf{e}} \tag{39}$$

$$\frac{\partial \delta \dot{z}}{\partial \Delta \mathbf{e}} = \frac{\partial \dot{r}_D}{\partial \Delta \mathbf{e}} \Delta z + \dot{r}_D \frac{\partial \Delta z}{\partial \Delta \mathbf{e}} + \frac{\partial r_D}{\partial \Delta \mathbf{e}} \Delta \dot{z} + r_D \frac{\partial \Delta z}{\partial \Delta \mathbf{e}} \tag{40}$$

RESULTS

To evaluate the proposed method, the predicted relative motion by the unit sphere STM is compared with those by the Gim-Alfriend STM.

<u>Chief and Deputy Orbits</u>

The mean elements of the Chief orbit are

$a = 8000\,(\text{km}),\ i = 50\,(\text{deg}),\ e = 0.01$

$\Omega = 0\,(\text{deg}),\ \omega = 0\ (\text{deg}),\ M_0 = 0\ (\text{deg})$

The Deputy orbit can be obtained [14]

$$\delta a = -0.5 J_2 a \left(\frac{R_e}{a}\right)^2 \left(\frac{3\eta + 4}{\eta^4}\right) \left[\left(1 - 3\cos^2 i\right) \left(\frac{q_1 \delta q_1 + q_2 \delta q_2}{\eta^2}\right) + \sin 2i \delta i \right] \tag{41}$$

$$\delta \lambda = -\delta \Omega \cos i \tag{42}$$

$$\delta i = \frac{\rho \cos \alpha_0}{a} \tag{43}$$

$$\delta q_1 = -\frac{\rho \sin \alpha_0}{2a} \tag{44}$$

$$\delta q_2 = -\frac{\rho \cos \alpha_0}{2a} \tag{45}$$

$$\delta \Omega = -\frac{\rho \sin \alpha_0}{a \sin i} \tag{46}$$

where $\lambda = \omega + M$, $\eta = \sqrt{1 - e^2}$. ρ is the relative orbit size and α_0 is the initial phase angel. Eqs(41-46) are used to establish the projected circular orbit (PCO) for a circular Chief orbit. Actually Eq.(41) is the bounded condition or period matching condition. When the Chief orbit is not circular they establish a relative motion orbit that is close to a PCO. Because the period matching condition is used there is very little in-track drift. The radius of the PCO ρ is chosen as 40 km. The initial phase angle α_0 is set to zero.

<u>Comparisons</u>

Figs. 1-4 show the position and velocity errors using the unit sphere STM, Gim-Alfriend STM. The errors are obtained by comparing the solutions from the STMs with the numerical integrations, respectively. The solutions of the numerical integrations are obtained by numerically integrating the equations of motion of both satellites in the ECI frame with a J_2 gravity field , differencing them and transforming the LVLH frame.

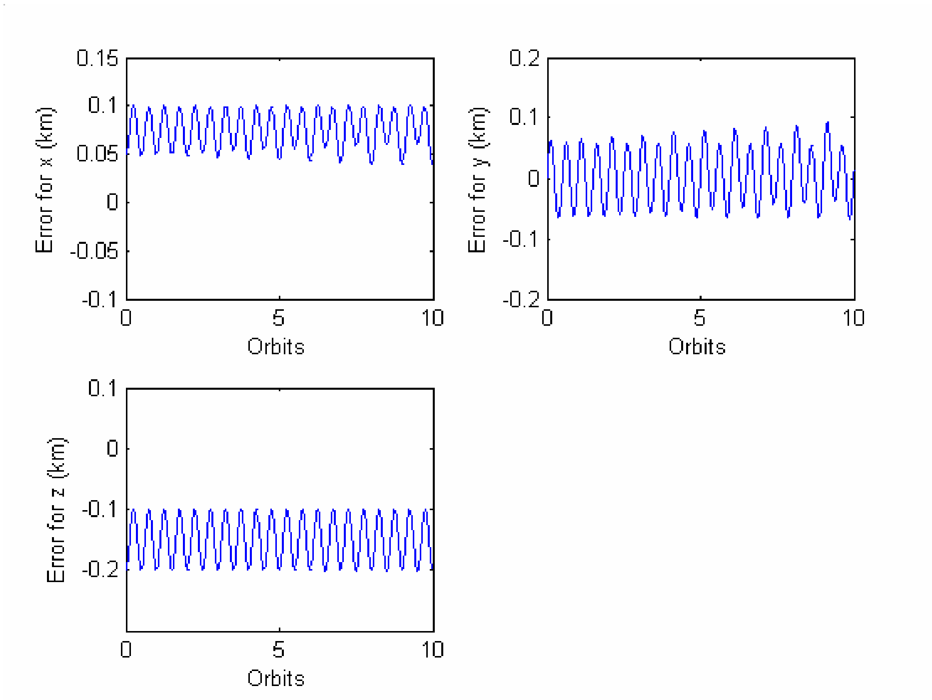

Fig.1 Position Errors by Unit Sphere STM

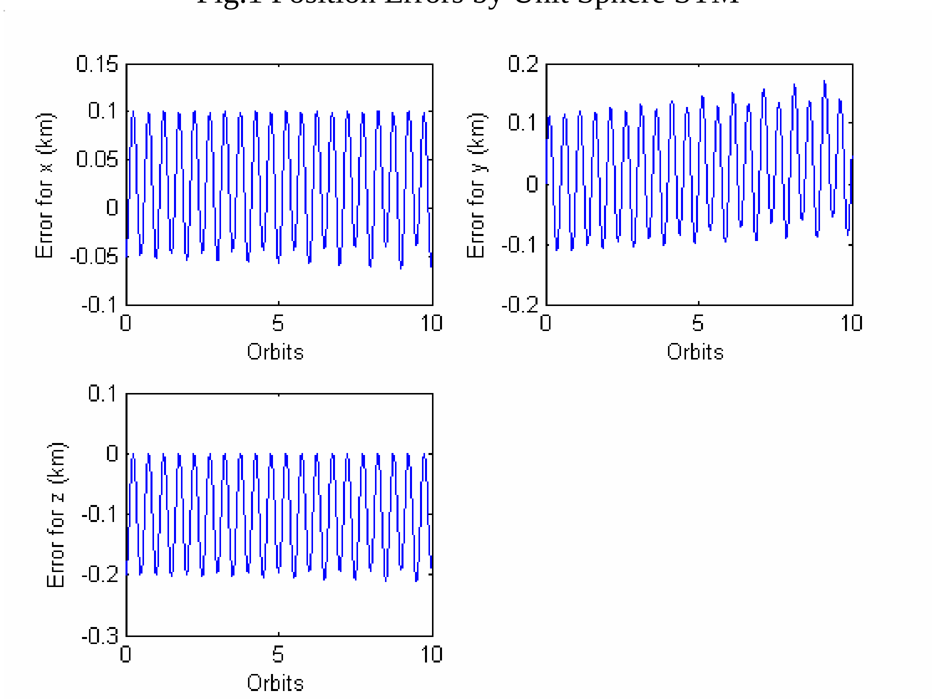

Fig.2 Position Errors by Gim-Alfriend STM

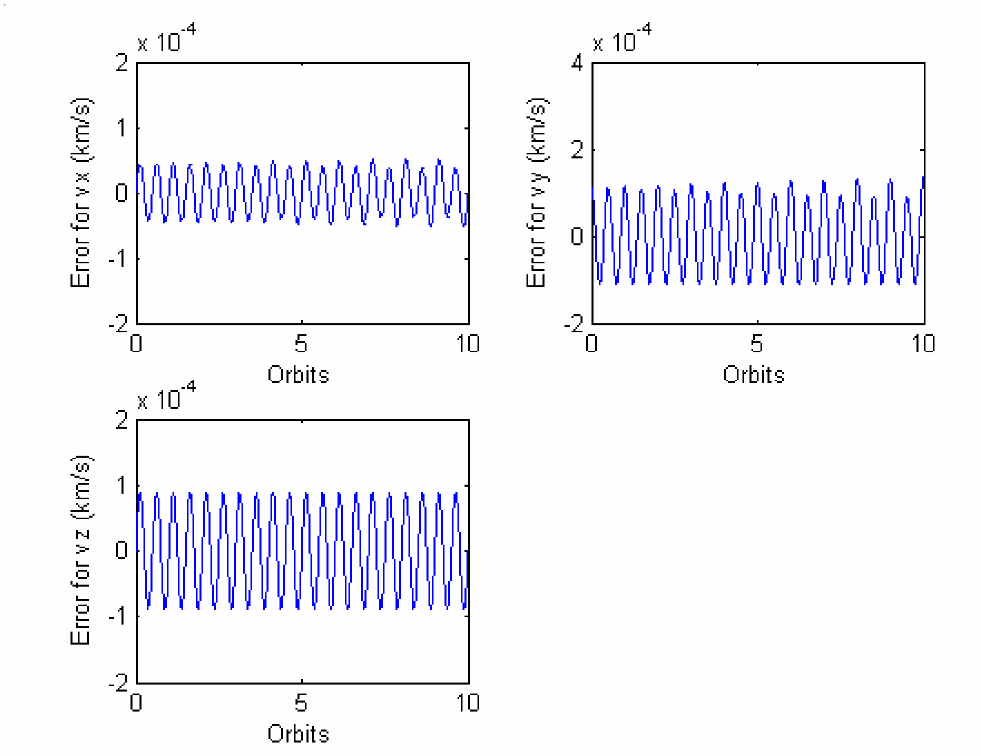

Fig.3 Velocity Errors by Unit Sphere STM

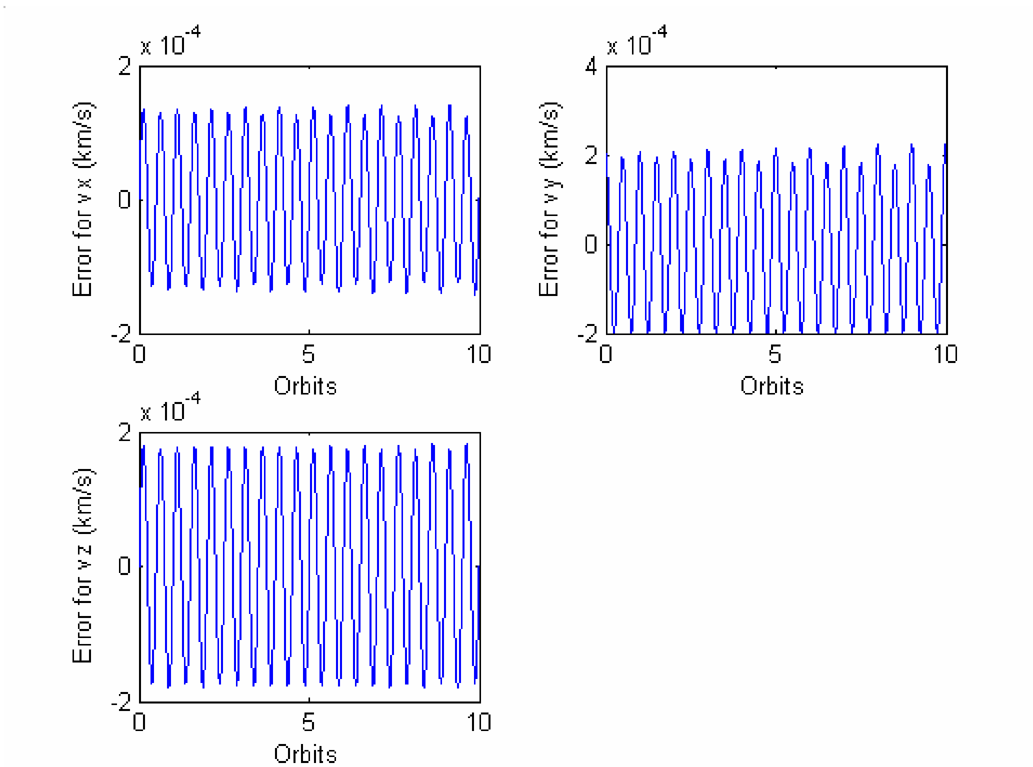

Fig.4 Velocity Errors by Gim-Alfriend STM

The results indicate the STMs are very accurate descirbing the relative motions. The STM from the unit sphere is better than that from Gim-Alfriend, since the former is achieved from the exact kinematic description, not from linear expressions in differential orbital elements to build the latter.

One can see there are initial biases in x and z directions and small drift in y direction in Figs. 1-2. The initial bias means the PCO is not centered. The small secular drifts are caused by neglecting J_2^2 in the mean elements propagation and using linear periodic matching condition Eq. (41).

Figs.1 and 3 have the same characteristics as Figs. 2 and 4 except the amplitudes. This is because we just rebuild the transformation matrix from the orbital elements to the coordinates in the LVLH in the Gim-Alfriend STM.

To get a big picture of the comparison, the results from the above methods are shown through the nonlinear modeling index in Fig. 5. The nonlinear index concept is developed by Alfriend and Yan [13] to compare the accuracy of the methods. The radius ρ of the PCO is chosen as 0.16, 0.80, 1.6, 4, 8, 12, 16, 40, 80, 120 and 160 km and the eccentricity of Chief orbit 0.01. Fig. 5 illustrates the index varying with the radius when the eccentricity is 0.1.

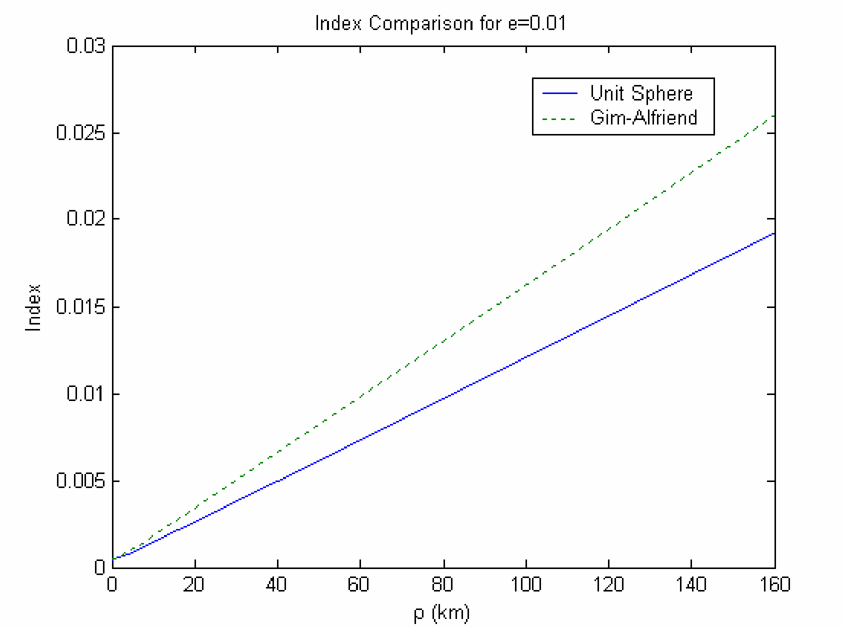

Fig. 5 Index Comparison for $e = 0.01$

The modeling error index is an effective tool for evaluating the accuracy of approximate methods of relative motions. Fig. 5 illustrates the indices grow linearly with increasing radius, since nonlinearity increases with the PCO size.

Fig. 5 again shows the result from the unit sphere approach is better than that from the Gim-Alfriend STM, although both of them provide a good representation of the motion.

CONCLUSIONS

We develop a high accurate STM to describe relative motions using the unit sphere approach. We use the error comparison and index comparison to compare it with the Gim-Alfriend STM. The result from the unit sphere STM is better than that from the Gim-Alfriend STM.

ACKNOWLEDGEMENT

This research was supported by NASA Goddard Space Flight Center under Grant No. NAG5-11349.We thank them for their support. We also thank the Graduate Student Research and Presentation for a travel grant funded by the Association of Former Students and Offices of Graduate Studies in Texas A&M University.

REFERENCE

1. Clohessy, W. H. and Wiltshire, R. S., "Terminal Guidance System for Satellite Rendezvous," *J. of the astronautical Sciences*, Vol. 27, No. 9, Sept. 1960, pp. 653-678
2. Hill, G. W., "Researches in the Lunar Theory," *American Journal of Mathematics*, Vol. 1, 1878, pp. 5-26.
3. Lawden, D. F., *Optimal Trajectories for space Navigation*, Butterworths, London, 1963.
4. Tschauner, J. and Hempel, P., "Optimale Beschleunigeungsprogramme fur das Rendezvous-Manover," *Astronautica Acta*, Vol. 10, 1964, pp. 296-307.
5. Carter, T. E., "New Form for the Optimal Rendezvous Equations Near a Keplerian Orbit," *Journal of Guidance, Control and Dynamics*, Vol. 13, No.1, Jan-Feb 1990.
6. Melton, R. G., "Time-Explicit Representation of Relative Motion Between Elliptical Orbits," *Journal of Guidance, Control and Dynamics*, Vol. 23, No.4, 2000, pp. 604-610.
7. Garrison, J. L., Gardner, T. G., and Axelrad, P., "Relative Motion in Highly Elliptical Orbits", Paper AAS-95-194, *AAS/AIAA Spaceflight Mechanics 1995*, Vol. 89, 1995, pp. 1359-1376.
8. Junkins, J. L., Akella, M. R., and Alfriend, K. T., "Non-Gaussian Error Propagation in Orbital Mechanics," *AAS J. of Astronautical Sciences*, Vol. 44, Oct.-Dec. 1996, pp. 541-563.
9. Alfriend K. T., Schaub, H., and Gim, D.-W, "Gravitational Perturbations, Nonlinearity and Circular Orbit Assumption Effects on Formation Flying Control Strategies" Paper AAS 00-012, AAS Guidance and Control 2000, Vol. 104, 2000, pp. 139-158

10. Gim, D.-W. and Alfriend, K. T., "State Transition Matrix of Relative Motion for the Perturbed Non-Circular Reference Orbit," *Journal of Guidance, Control and Dynamics*, Vol. 26, No.6, 2003, pp. 956-971.
11. Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," *The Astronomical Journal*, Vol. 64, No. 1274, 1959, pp. 378-397
12. Vadali, S. R., "An Analytical Solution for Relative Motion of Satellites," DCSSS Conference, Cranfield University, Cranfield, UK, July 2002
13. Alfriend, K. T., Yan, H., "An Evaluation and Comparison of Relative Motion Theories," Paper AAS 03-567, AAS/AIAA Astrodynamics Specialists Conference, Big Sky, Montana, August 3-7, 2003
14. Vaddi, S. S., Alfriend, K. T., and Vadali, S. R., "Sub-Optimal Formation Establishment and Reconfiguration using Impulsive Control," paper AAS 03-590, AAS/*AIAA Astrodynamics Specialist Conference*, Big Sky, Montana, August 3-7, 2003.